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Analyticity of the Planar Limit of a Matrix Model

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Abstract. Using Chebyshev polynomials combined with some mild combinatorics, we provide an alternative approach to the analytical and formal planar limits of a random matrix model with a 1-cut potential V. For potentials $V(x) = x^2/2 - \sum_{n\geq 1} a_n x^n/n$, as a power series in all a_n, the formal Taylor expansion of the analytic planar limit is exactly the formal planar limit. In the case V is analytic in infinitely many variables $\{a_n\}_{n\geq 1}$ (on the appropriate spaces), the planar limit is also an analytic function in infinitely many variables and we give quantitative versions of where this is defined. Particularly useful in enumerative combinatorics are the gradings of $V, V_t(x) = x^2/2 - \sum_{n\geq 1} a_n t^{n/2} x^n/n$ and $V_t(x) = x^2/2 - \sum_{n\geq 3} a_n t^{n/2-1} x^n/n$. The associated planar limits $F(t)$ as functions of t count planar diagram sorted by the number of edges respectively faces. We point out a method of computing the asymptotic of the coefficients of $F(t)$ using the combination of the *wzb* method and the resolution of singularities. This is illustrated in several computations revolving around the important extreme potential $V_t(x) = x^2/2 + \log(1 - \sqrt{t}x)$ and its variants. This particular example gives a quantitative and sharp answer to a conjecture of 't Hooft's, which states that if the potential is analytic, the planar limit is also analytic.

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1. Introduction

1.1. Formal Matrix Models and Their Planar Limit

Matrix models are integrals of exponentiated potential functions over finite dimensional vector spaces (such as the vector space of Hermitian matrices

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of size N) that were studied in the seventies as an approximation of Quantum Field Theory in a 0-dimensional space-time. Matrix models at fixed value of N and their behavior when N is large is useful in a variety of problems that include enumerative problems of ribbon graphs, random two-dimensional gravity, triangulations of surfaces, random matrices, topological string theory, intersection theory on the moduli space of curves and perturbative gauge theory; see [7,11,13,14,29–31,38].

Matrix models come in two flavors: formal and analytic. *Formal matrix models* (FMM in short) are easy to define, using formal Gaussian integration. The input of a formal matrix model is a *formal potential* $\mathcal{V}$

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n=1}^{\infty} \frac{a_n}{n} x^n \in \mathcal{A}[[x]], \tag{1}$$

which lies in a formal power series ring $\mathcal{A}[[x]]$, where $\mathcal{A}$ is the completed ring

$$\mathcal{A} = \mathbb{Q}[[a_1, a_2, a_3, \ldots]]. \tag{2}$$

The *partition function* $\mathcal{Z}$ and the *free energy* $\mathcal{F}$ of the formal matrix model is given by the following formal integral and its logarithm, respectively

$$\mathcal{Z} = \frac{\int_{\mathcal{H}_N} \mathrm{d}M \exp(-N\mathrm{Tr}(\mathcal{V}(M)))}{\int_{\mathcal{H}_N} \mathrm{d}M \exp(-N\mathrm{Tr}(M^2/2))}, \quad \mathcal{F} = \log \mathcal{Z} \in N^2\mathcal{A}[[1/N^2]] \tag{3}$$

where

- $\mathcal{H}_N$ is the vector space of Hermitian matrices of size N,
- $\mathrm{Tr}(M)$ denotes the trace of a matrix M,
- The meaning of the formal integration is the following: expand $e^{-N\mathrm{Tr}(\mathcal{V}(M)+M^2/2)}$ as formal power series in $\mathcal{A}[[N, \mathrm{Tr}(M), \mathrm{Tr}(M^2), \ldots]]$ and integrate coefficient-wise. This operation produces a well-defined element of $N^2\mathcal{A}[[1/N^2]]$.

So, we can write

$$\mathcal{F} = \sum_{g=0}^{\infty} N^{2-2g}\mathcal{F}_g, \quad \mathcal{F}_g \in \mathcal{A}. \tag{4}$$

$\mathcal{F}_g \in \mathcal{A}$ is called the *genus g-limit* of the formal matrix model. We can expand $\mathcal{F}_g$ in terms of monomials

$$\mathcal{F}_g = \sum_{\lambda} c_{\lambda,g} a_{\lambda}$$

where the sum is over the set of all *partitions* $\lambda = (1^{n_1} 2^{n_2} \cdots)$, and $a_\lambda = \prod_j a_j^{\lambda_j}$ and c_λ are rational numbers. $\mathcal{F}_g$ enumerates connected ribbon graphs of arbitrary valency on a connected, oriented surface of genus g; see [6,7,28,33]. More precisely, it follows by *Wick's theorem* that c_λ is the weighted sum of all connected ribbon graphs (weighted by the inverse of the order of the automorphism group) of genus g that have n_k vertices of valency k; see [28,33]. When $g = 0, \mathcal{F}_0$ is the *planar limit* of the formal matrix model. The planar limit depends on the formal potential $\mathcal{V}$, and if we want to stress this dependence, we will use the notation $\mathcal{F}_{0,\mathcal{V}}$. As an example, when

$$\mathcal{V}_4 = \frac{x^2}{2} - \frac{a_4}{4}x^4$$

the coefficients of the formal power series $\mathcal{F}_{0,\mathcal{V}_4} \in \mathbb{Q}[[a_4]]$ counts the weighted sum of connected planar 4-valent ribbon graphs. From the definition of $\mathcal{F}_{0,\mathcal{V}_4}$, one can compute several terms of the power series $\mathcal{F}_{0,\mathcal{V}_4}$, by hand or by machine. The pioneering work of [6,7] gave an exact formula for the power series $\mathcal{F}_{0,\mathcal{V}_4}$ using *potential theory*:

$$\mathcal{F}_{0,\mathcal{V}_4} = \frac{1-36a_4+162a_4^2+(1-30a_4)\sqrt{1-12a_4}}{432a_4^2} + \frac{1}{2}\log\left(\frac{1-\sqrt{1-12a_4}}{6a_4}\right) \in \mathbb{Q}[[a_4]]$$

The computation of [6,7] was lacking rigor, and several years later their method was justified by using potential theory and the Riemann–Hilbert method; see [12,16]. In the present paper, we give an independent proof, using mostly techniques from real analysis and elementary potential theory. In addition, we describe explicitly the analyticity properties of $\mathcal{F}_0$ with sharp results, see Theorems 1.1 and 1.2 below.

As a notational convention, we will use caligraphic symbols $\mathcal{V}, \mathcal{R}, \mathcal{S}, \mathcal{F}_0, \ldots$ for formal matrix models and straight symbols as $V, R, S, F_0, \ldots$ for the analytic matrix models.

1.2. Analytic Matrix Models and Their Planar Limit

Let us now define the *analytic matrix models* (AMM in short). An *admissible potential* $V(x)$ is a function $V : \mathbb{R} \longrightarrow \mathbb{R}$ which is lower-semicontinuous, and grows sufficiently at infinity, i.e., satisfies

$$\lim_{|x|\to\infty} \frac{V(x)}{2\log|x|} > 1. \tag{5}$$

For an analytic matrix model with an admissible potential V define

$$\begin{aligned} I_{\mathrm{V}} &= -\lim_{N\to\infty}\frac{1}{N^2}\log\int_{\mathcal{H}_N}\exp(-N\mathrm{Tr}(V(M)))\mathrm{d}M \\ &= \inf_{\mu\in\mathcal{P}(\mathbb{R})}\left\{\int V(x)\mu(\mathrm{d}x) - \iint \log|x-y|\mu(\mathrm{d}x)\mu(\mathrm{d}y)\right\}, \end{aligned} \tag{6}$$

where $\mathcal{P}(\mathbb{R})$ is the set of all probability measures on $\mathbb{R}$. The second equality in the above equation follows for example from [10,23].

In the case $V(x) = \frac{x^2}{2} - \sum_{n\geq 1}\frac{a_n x^n}{n}$ is an admissible potential we then define the *analytic planar limit* as

$$F_{0,\mathrm{V}} = \frac{3}{4} - I_{\mathrm{V}}. \tag{7}$$

We will call $F_{0,\mathrm{V}}$ and I_{V} the *the analytic planar limit.*

Figure 1. A graphical interpretation of b, c and R, S in terms of the endpoints of the support of the equilibrium measure

As we already mentioned, this formula allows one to reduce the problem of the planar limit to the investigation of what is known in the literature as the logarithmic potential with external fields (Fig. 1).

A *1-cut potential* V is an admissible potential whose equilibrium measure has support in a *single interval* $[b-2c, b+2c]$. Following the notation of [4], we will use the change of variables (see Fig. 1 for a graphical representation)

$$(b, c^2) = (S, R). \tag{8}$$

It turns out that in the case V is a 1-cut potential (plus some nondegeneracy), and V_a is an analytic perturbation of V, then the endpoints and the planar limit depend analytically on a.

An admissible potential V is *even* if it satisfies $V(x) = V(-x)$ for all $x \in \mathbb{R}$. For even 1-cut potentials, the equilibrium measure of V is centered at $b = 0$.

1.3. Analyticity of The Planar Limit

Analyticity of functions in infinitely many variables is well defined and understood on functions defined on ℓ^1 spaces (see [26] and [36]). In our case we need to define a weighted version of ℓ^1 space. To this end, let $r > 0$ be a positive number, and for a complex-valued sequence $\mathbf{a} = \{a_n\}_{n\geq 1} \subset \mathbb{C}^{\mathbb{N}}$, consider its ℓ^1_r norm

$$||\mathbf{a}||_r = \sum_{n=1}^{\infty} |a_n| r^n. \tag{9}$$

Now, consider the following ℓ^1 type spaces

$$\ell^1_r(\mathbb{N}) = \{\mathbf{a} = \{a_n\}_{n\geq 1} \subset \mathbb{C}^{\mathbb{N}} : ||\mathbf{a}||_r < \infty\}$$
$$\ell^1_r(2\mathbb{N}) = \{\mathbf{a} = \{a_n\}_{n\geq 1} \in \ell^1_r(\mathbb{N}) : a_{2n} = 0, n \geq 1\}.$$

Let B_r and B_r^{ev} denote the open balls of radius 1 in $\ell^1_r(\mathbb{N})$ and $\ell^1_r(2\mathbb{N})$, respectively.

Now consider $\mathfrak{S} \subset \mathbb{R}^{\mathbb{N}}$ to be the set of sequences $\mathbf{a} = \{a_n\}_{n\geq 1} \in \mathbb{R}^{\mathbb{N}}$ such that

$$V(x) = \frac{x^2}{2} - \sum_{n\geq 1} \frac{a_n x^n}{n}$$

is a 1-cut admissible potential which is analytic near 0. Using Eq. (7) we can define a map $F_{0,\mathrm{V}}$

$$F_0 : \mathfrak{S} \longrightarrow \mathbb{R}. \tag{10}$$

Likewise, we have a map $F_0^{\mathrm{ev}} : \mathfrak{S}^{\mathrm{ev}} \longrightarrow \mathbb{R}$.

We use here the definition of [26,36] for an analytic function on $\ell^1_r(\mathbb{N})$ which essentially means that the Taylor series in infinitely many variables converges.

Theorem 1.1. *The maps F_0, R and S (resp. $F_0^{\mathrm{ev}}, R^{\mathrm{ev}}$ and S^{ev}) uniquely extend to analytic functions on $B_{1/\sqrt{12}}$ (resp. $B^{\mathrm{ev}}_{1/\sqrt{8}}$).*

Our next theorem identifies the planar limit of the formal and analytic matrix model. Since the map F_0 from (10) is analytic at $0 \in \ell^1_r(\mathbb{N})$, its Taylor series regarded as a formal power series is given by

$$F_0 = \sum_\lambda c_\lambda a_\lambda \in \mathcal{A} \tag{11}$$

where the sum is over the set of partitions (including the empty one), $c_\lambda \in \mathbb{Q}$ and $\mathcal{A}$ is given in (2). Consider the formal power series $(\mathcal{R}, \mathcal{S}) \in \mathcal{A}^2$ defined in Sect. 1.4 below.

Theorem 1.2. *We have*

$$R = \mathcal{R}, \quad S = \mathcal{S}, \quad F_0 = \mathcal{F}_0. \tag{12}$$

What this means is that, if the analytical procedures are taken formally, one recaptures the planar limit of the formal matrix models.

Theorems 1.1 and 1.2 confirm a conjecture of 't Hooft for the planar limit of matrix models. 't Hooft's conjecture is motivated by perturbative gauge theory ideas whose Feynman diagrams are ribbon graphs, and asserts that $\mathcal{F}_0(\mathcal{V}(x))$ should be an analytic function at $x = 0$ when $\mathcal{V}(x)$ is analytic at zero; [38]. For a proof of 't Hooft's conjecture for the case of Chern–Simons gauge theory, see [21].

A natural problem is to extend Theorem 1.1 to all genera g.

Problem 1.1. Show that for all $g \geq 0, \mathcal{F}_g$ (resp., $\mathcal{F}_g^{\mathrm{ev}}$) is the Taylor series of an analytic function on $B_{1/\sqrt{12}}$ (resp. $B_{1/\sqrt{8}}$).

This may be achieved using [1,18].

1.4. Two Gradings for The Planar Limit

The formal planar limit $\mathcal{F}_0 \in \mathcal{A}$ enumerates planar ribbon graphs of arbitrary valency, and it is closely related to two other formal power series $(\mathcal{R}, \mathcal{S})$ which are uniquely determined by the system of non-linear equations

$$\begin{cases} \mathcal{R} = \mathcal{H}_1(\mathcal{R}, \mathcal{S}) \\ \mathcal{S} = \mathcal{H}_2(\mathcal{R}, \mathcal{S}) \end{cases} \tag{13}$$

where

$$\mathcal{H}_1(\mathcal{R}, \mathcal{S}) = 1 + \sum_{n \geq 1} a_n \sum_{j \geq 1} \binom{n-1}{j-1} \binom{n-j}{j} \mathcal{R}^j \mathcal{S}^{n-2j} \tag{14}$$

$$\mathcal{H}_2(\mathcal{R}, \mathcal{S}) = \sum_{n \geq 1} a_n \sum_{j \geq 0} \binom{n-1}{2j} \binom{2j}{j} \mathcal{R}^j \mathcal{S}^{n-2j-1} \tag{15}$$

Equation (13) always has a unique solution in $(\mathcal{R}, \mathcal{S}) \in \mathcal{A}^2$ that satisfies $R \in 1 + \mathcal{A}^+$ and $S \in \mathcal{A}^+$, where $\mathcal{A}^+$ are the formal power series in the variables a_n with no constant term. Moreover, it is easy to see that this unique formal solution has integer coefficients.

An enumerative interpretation of the coefficients of $(\mathcal{R}, \mathcal{S})$ is given in [4], which in particular implies that they are natural numbers. An analytic interpretation of $(\mathcal{R}, \mathcal{S})$ is that they determine the position of the interval of a 1-cut analytic matrix model; see Sect. 6.

Enumerative combinatorics dictates two gradings on the set of variables a_n, the *edge grading* $\deg_{\mathrm{e}}(a_n)$ and the *face grading* $\deg_{\mathrm{f}}(a_n)$ defined by

$$\deg_{\mathrm{e}}(a_n) = \frac{n}{2}, \quad \deg_{\mathrm{f}}(a_n) = \frac{n}{2} - 1. \tag{16}$$

Given an element $\mathcal{H} \in \mathcal{A}$, let $\mathcal{H}_{\mathrm{e}} \in \mathcal{A}[[t^{1/2}]]$ and $\mathcal{H}_f$ denote the result of substituting a_n by $a_n t^{n/2}$ and $a_n t^{n/2-1}$ respectively. For example, for the formal potential $\mathcal{V}(x)$ from Eq. (1) we have

$$\mathcal{V}_{\mathrm{e}}(x) = \frac{x^2}{2} - \sum_{n=1}^{\infty} \frac{a_n}{n} t^{n/2} x^n \in \mathcal{A}[[t^{1/2}, x]],$$

$$\mathcal{V}_{\mathrm{f}}(x) = \frac{x^2}{2} - \sum_{n=3}^{\infty} \frac{a_n}{n} t^{n/2-1} x^n \in \mathcal{A}[[t^{1/2}, x]] \tag{17}$$

where in the latter we assume that $a_1 = a_2 = 0$. Likewise, for $\mathcal{R}_{\mathrm{e}}, \mathcal{R}_{\mathrm{e}}$ and $\mathcal{F}_{0,\mathrm{e}}$.

Of course, when we set $t = 1$ to $\mathcal{H}_{\mathrm{e}}$ or $\mathcal{H}_{\mathrm{f}}$, we recover $\mathcal{H}$. In particular,

$$\mathcal{F}_{0,\mathrm{e}}(1) = \mathcal{F}_{0,\mathrm{f}}(1) = \mathcal{F}_0 \in \mathcal{A} \tag{18}$$

The next theorem gives a simple formula for $\mathcal{F}_{0,\mathrm{e}}$ in terms of $\mathcal{R}_{\mathrm{e}}$ and $\mathcal{S}_{\mathrm{e}}$. This appears in [5] but for polynomial potentials $\mathcal{V}$ and the proof in there uses orthogonal polynomials.

Theorem 1.3. *We have:*

$$\mathcal{F}_{0,\mathrm{e}}(t) = \frac{1}{t} \int_0^t \frac{(t-s)(2\mathcal{R}_{\mathrm{e}}(s)\mathcal{S}_{\mathrm{e}}^2(s) + \mathcal{R}_{\mathrm{e}}^2(s) - 1)}{2s} \, \mathrm{d}s. \tag{19}$$

It follows that

$$(t^2 \mathcal{F}_{0,\mathrm{e}}')' = \frac{2\mathcal{R}_{\mathrm{e}}(t)\mathcal{S}_{\mathrm{e}}^2(t) + \mathcal{R}_{\mathrm{e}}^2(t)}{2}. \tag{20}$$

where f' indicates the derivative with respect to t.

The next theorem gives a simple formula for $\mathcal{F}_{0,\mathrm{f}}$ in terms of $\mathcal{R}_{\mathrm{f}}$ alone.

Theorem 1.4. *We have:*

$$\mathcal{F}_{0,\mathrm{f}}(t) = \frac{1}{t^2} \int_0^t (t-s) \log \mathcal{R}_{\mathrm{f}}(s) \mathrm{d}s \tag{21}$$

In particular

$$(t^2 \mathcal{F}_{0,\mathrm{f}})'' = \log \mathcal{R}_{\mathrm{f}}(t). \tag{22}$$

Remark 1.2. Given a potential $\mathcal{V} = \sum_{n\geq 1} a_n x^n/n$ as a formal power series, from the potential theoretic approach one obtains that at the formal level, c and b satisfy the system

$$\begin{aligned} 2 &= \int_{-2}^{2} cx\mathcal{V}'(cx+b)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = \sum_{n\geq 1} a_n \sum_{j\geq 1} \binom{n-1}{2j-1}\binom{2j}{j} c^{2j} b^{n-2j} \\ 0 &= \int_{-2}^{2} \mathcal{V}'(cx+b)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = \sum_{n\geq 1} a_n \sum_{j\geq 0} \binom{n-1}{2j}\binom{2j}{j} c^{2j} b^{n-2j-1}. \end{aligned} \tag{23}$$

In the case $\mathcal{V} = x^2/2 - \sum_{n\geq 1} a_n x^n/n$, and $R = c^2, S = b$, one easily obtains the system (13).

Remark 1.3. Some authors prefer to consider the following rescaling $\widetilde{\mathcal{V}}_{\mathrm{e}}$ of $\mathcal{V}_{\mathrm{e}}$

$$\widetilde{\mathcal{V}}_{\mathrm{e}} = \frac{x^2}{2t} - \sum_{n\geq 1} \frac{a_n x^n}{n}$$

Since $\mathcal{V}_{\mathrm{e}}(x) = \widetilde{\mathcal{V}}_{\mathrm{e}}(t^{1/2}x)$, it is easy to see that

$$\begin{aligned} \tilde{c}(t) &= \sqrt{t}c(t) \\ \tilde{b}(t) &= \sqrt{t}b(t), \end{aligned}$$

where $\tilde{c}(t)$ and $\tilde{b}(t)$ are defined using the system (23).

Remark 1.4. Likewise, for the following rescaling $\widetilde{\mathcal{V}}_{\mathrm{f}}$ of $\mathcal{V}_{\mathrm{f}}$

$$\widetilde{\mathcal{V}}_{\mathrm{f}} = \frac{x^2}{2} - \sum_{n\geq 3} \frac{a_n x^n}{n}$$

we have $\mathcal{V}_{\mathrm{f}}(x) = \widetilde{\mathcal{V}}_{\mathrm{f}}(t^{1/2}x)/t$ which implies that [here $\tilde{c}(t)$ and $\tilde{b}(t)$ are constructed from $\tilde{\mathcal{V}}(x)/t$]

$$\begin{aligned} \tilde{c}(t) &= \sqrt{t}c(t) \\ \tilde{b}(t) &= \sqrt{t}b(t). \end{aligned}$$

Remark 1.5. When $\mathcal{V}$ is even, then $\mathcal{S} = 0$ and $\mathcal{R}$ satisfies the implicit equation

$$\mathcal{H}(\mathcal{R}) = 1 \tag{24}$$

where

$$\mathcal{H}(x) = x - \frac{1}{2}\sum_{n=3}^{\infty} a_{2n} x^n \binom{2n}{n} = x - \sum_{n=3}^{\infty} a_{2n} x^n \binom{2n}{n-1}. \tag{25}$$

This is indeed so because

$$\int_{0}^{\pi} \frac{x\mathcal{V}'(xy)\mathrm{d}y}{t\pi\sqrt{4-y^2}} = \frac{1}{t}\left(x - \frac{1}{2}\sum_{n=3}^{\infty} a_{2n} x^{2n}\binom{2n}{n}\right).$$

1.5. Algebricity, Holonomicity and Asymptotics of The Planar Limit

In this section we discuss the algebricity of the planar limit. Let us recall first some well-known properties of algebraic functions and the asymptotics of their Taylor coefficients. The reader may consult [39] and also [19, Chpt.VII] for further details. Computer implementations are available at [15,24,34].

An *algebraic function* $y = y(x)$ is one that satisfies a polynomial equation $P(y, x) = 0$ for some 2-variable polynomial with rational coefficients. Below, we will be interested in algebraic functions $y(x)$ which are *regular* at $x = 0$, i.e., they have a Taylor series expansion

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{26}$$

The set of algebraic functions is a field, closed under differentiation with respect to x. Algebraic functions are always *holonomic*, i.e., they satisfy (regular singular) linear differential equations with coefficients polynomials in x with rational coefficients. An algebraic function $y(x)$ gives rise to a ramified d-sheeted covering $\mathbb{C} \longrightarrow \mathbb{C}$ with semisimple local monodromy (with eigenvalues complex roots of unity) and global monodromy a *finite* subgroup of $\mathrm{SL}(d, \mathbb{C})$. In other words, an algebraic function $y(x)$ regular at $x = 0$ can be uniquely analytically continued as a multivalued analytic function on $\mathbb{C} \setminus \Lambda$, where Λ is a finite set of algebraic numbers. In practice the analytic continuation can be obtained via *Puiseux series*, and all local expansions of $y(x)$ around a singularity $x \in \Lambda$ are exactly computed by $y(x)$; see for example [15,34]. Since $y(x)$ is holonomic, it follows that the sequence (a_n) of its Taylor coefficients from (26) is *holonomic*, i.e., it satisfies a linear difference equation with coefficients polynomials in n with rational coefficients; see [41]. To discuss the asymptotics of (a_n) we need to recall what is a sequence of Nilsson type, discussed in detail in [20].

Definition 1.6. We say that a sequence (a_n) is of Nilsson type if it has an asymptotic expansion of the form:

$$a_n \sim_{n \to \infty} \sum_{\lambda,\alpha,\beta} \lambda^n n^\alpha (\log n)^\beta S_{\lambda,\alpha,\beta} h_{\lambda,\alpha,\beta}\left(\frac{1}{n}\right) \tag{27}$$

where

- the summation is over a finite set,
- the *growth rates* λ are algebraic numbers of equal modulus,
- the *exponents* α are rational and the *nilpotency exponents* β are natural numbers,
- the *Stokes constants* $S_{\lambda,\alpha,\beta}$ are complex numbers,
- the formal power series $h_{\lambda,\alpha,\beta}(x) \in K[[x]]$ are Gevrey-1 (i.e., the coefficient of x^n is bounded by $C^n n!$ for some $C > 0$),
- K is a *number field* generated by the coefficients of $h_{\lambda,\alpha,\beta}(x)$ for all λ, α, β.

For a detailed discussion of the uniqueness, existence and computation of the asymptotic expansion of a sequence (a_n) of Nilsson type, see [20]. The results of [20] and the above discussion implies the following.

Proposition 1.7. (a) *If $y(x)$ is algebraic and regular at $x = 0$, then the sequence (a_n) defined by (26) is of Nilsson type, where $\beta = 0$ in (27).*

(b) *Moreover, the asymptotic expansion (27) can be computed exactly and effectively.*

We will apply the above proposition to the planar limit.

Proposition 1.8. (a) *If $\mathcal{R}_{\mathrm{e}}(t), \mathcal{S}_{\mathrm{e}}(t)$ (resp. $\mathcal{R}_{\mathrm{f}}(t)$) are algebraic functions, then $(t^2\mathcal{F}'_{0,\mathrm{e}})'(t)$ (resp. $(t^2\mathcal{F}_{0,\mathrm{f}})'''(t)$) is also an algebraic function.*

(b) *If $\mathcal{V}$ is a polynomial, then $\mathcal{R}_{\mathrm{e}}, \mathcal{S}_{\mathrm{e}}$ and $\mathcal{R}_{\mathrm{f}}$ are algebraic functions.*

(c) *Let*

$$\mathcal{F}_0(t) = \sum_{n\geq 1} f_n t^n.$$

Under the assumptions (a) it follows that the sequence (f_n) is holonomic.

(d) *In addition, the sequence (f_n) is of Nilsson type.*

Several illustrations of the above proposition to extreme potentials are given in Sects. 8 and 9.

1.6. The Plan of The Paper

In Sect. 2 we introduce and discuss the formal matrix models with the two important gradings, the edge and the face gradings. Section 3 introduces the potential theoretic part of analytic matrix models and the preliminary results needed in Sect. 4 where the main analytic results are presented. We use here real analysis tools combined with Chebyshev polynomials and elementary combinatorics to deal with the minimization problem (7), which is an alternative to the classical complex analysis techniques.

Section 4 is the bulk of the analysis, the central pieces being Theorems 4.2 and 4.3. These are applied to some analytic examples in Sect. 5.

Next, in Sect. 6 we prove the matching claimed in Theorem 1.2 and in Sect. 7 we give the proofs of the main results for the formal matrix models, namely, Theorems 1.3 and 1.4.

The main calculations with the extreme potentials are in Sects. 8 and 9, for the edge grading and respective the face grading. These main calculations are complemented with a small discussion in Sect. 10 about the calculations in the case of planar diagrams with vertices of valence 3 or 4.

In Sect. 11 we give the formal proof of 't Hooft's conjecture, materialized first in the general form of Theorem 11.1 and then in Corollary 11.1, from which Theorem 1.1 follows.

At last, Sect. 12 gives a perturbation result which is used in the proof of Theorem 1.2 in Sect. 6, though the results in this section do not give sharp results about the radius of convergence for the planar limit as in Sect. 11. However this is a very useful analytic tool and we decided to include here.

Finally, the appendix contains some Taylor series of $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{F}$. Some of these terms are used in the proof of the main results, Theorems 1.3 and 1.4.

2. Formal Matrix Models

It follows from the definition of the formal matrix model that the planar limit $\mathcal{F}_0$ is the generating series of counting of planar graphs, weighted by the inverse of the size of their automorphism groups. This is discussed in detail in [6, 17, 29, 33]. In particular, $\mathcal{F}_{0,\mathrm{e}}$ counts planar graphs where every n-valent edge contributes a term $t^{n/2}$. Likewise, $\mathcal{F}_{0,\mathrm{e}}$ counts planar graphs where every n-valent face contributes a term $t^{n/2-1}$.

3. Analytic Matrix Models

3.1. A Summary of Analytic Matrix Models

One of the main problems one faces with the minimization problem (7) is the support of the equilibrium measure. Without extra assumptions on the potential V, the support can be an arbitrary compact subset of the reals. However, most of the formal computations on the planar limit as a counting object are based on the formal manipulations as if the support was one interval.

Naturally, what we want to do here in the first place, is a complete analytical characterization of the one interval support for the equilibrium measure of (7). The way we do this here is based on an elementary approach to the logarithmic potential due to the following formula for $x, y \in [-2, 2]$:

$$\log|x-y| = -\sum_{n=1}^{\infty} \frac{2}{n} T_n\left(\frac{x}{2}\right) T_n\left(\frac{y}{2}\right)$$

where T_n are the the Chebyshev polynomials of first kind. Based on this formula we give a quick incursion into various formulae in logarithmic potential theory on $[-2, 2]$, especially the formula from Theorem 4.1 and show that the general case of 1-cut potentials can always be reduced by rescaling and translation in the x-variable to this case. The reason of doing this is to highlight a way of using manipulations of the Chebyshev polynomials in this framework. This seems to be an alternative (in the case of measures with support $[-2, 2]$) to the powerful complex analysis methods discussed for example in [37].

However, the more interesting fact is that we obtain the following explicit formula for I_{V}. If V is a C^3 potential whose equilibrium measure has support $[-2c + b, 2c + b]$, then

$$I_{\mathrm{V}} = -\log c + \int_{-2}^{2} \frac{V(cx+b)\mathrm{d}x}{\pi\sqrt{4-x^2}} - \int_0^c s\left[\left(\int_{-2}^{2} \frac{xV'(sx+b)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)^2 + \left(\int_{-2}^{2} \frac{V'(sx+b)\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)^2\right]\mathrm{d}s. \tag{28}$$

This is attained via concrete exploitations of the Chebyshev polynomials, first for the case of the interval $[-2, 2]$ and then simple rescaling. It is worth pointing out that ultimately, this identity reduces to checking a combinatorial identity for binomial coefficients. This we carry out using the implementation of the Zeilberger method.

The previous formula, makes the dependence on the potential very transparent. Any questions on the analyticity of I_{V} (or $F_{0,\mathrm{V}}$) under perturbation follows from the analyticity of the endpoints of the support of the equilibrium measure.

Finally, we show that under certain non-degeneracy conditions made explicit in Sect. 12, if $V_{\mathbf{t}}$ is an analytic perturbation of V depending on the parameter $\mathbf{t}$, then the planar limit $I_{\mathrm{V}_{\mathbf{t}}}$ depends analytically on $\mathbf{t}$ on a domain of the parameter space.

Here is an outline of what follows. In Sect. 3.2 we introduce the main objects, in Sect. 3.3 we discuss the formula that connects the logarithmic potentials and the Chebyshev polynomials. Next, in Sect. 3.4, we describe the connection with Fourier analysis. Section 4 contains the main analytical results.

3.2. Logarithmic Potentials with External Fields

As it was pointed out in the Introduction, we are going to look at the problem of minimizing the logarithmic energy with external fields and then investigate the planar limit in this framework.

Assume $V:\mathbb{R}\to\mathbb{R}$ is an admissible potential. For a closed set $S\subset\mathbb{R}$, according to [37] for the general case or [10] for the case $S=\mathbb{R}$, the following minimization problem has a unique solution (which turns out to be compactly supported)

$$I_{\mathrm{V}}(S)=\inf\{I_{\mathrm{V}}(\mu):\mu\in\mathcal{P}(S)\} \tag{29}$$

where $\mathcal{P}(S)$ stands for the set of probability measure on S and

$$I_{\mathrm{V}}(\mu)=\int V(x)\mu(\mathrm{d}x)-\iint\log|x-y|\mu(\mathrm{d}x)\mu(\mathrm{d}y). \tag{30}$$

The term $-\iint\log|x-y|\mu(\mathrm{d}x)\mu(\mathrm{d}y)$ is called the *logarithmic energy* of the measure μ. For simplicity, we will denote $I_{\mathrm{V}}=I_{\mathrm{V}}(\mathbb{R})$. Also for a given measure μ, we will denote $\operatorname{supp}\mu$, the support of the measure μ. The equilibrium measure of (29) on the set S (cf. [37, Thm.I.1.3]) is characterized by

$$\begin{aligned} V(x)&\geq 2\int\log|x-y|\mu(\mathrm{d}y)+C\quad\text{quasi-everywhere on } S\\ V(x)&=2\int\log|x-y|\mu(\mathrm{d}y)+C\quad\text{quasi-everywhere on } \operatorname{supp}\mu. \end{aligned} \tag{31}$$

Here, a property P holds "quasi everywhere" on the set Ω if we can find a set A such that $\mu(A)=0$ for any measure μ of finite logarithmic energy and the property P holds on $\Omega\backslash A$. This means, that the equality on $\operatorname{supp}\mu$ is almost surely realized with respect to any measure of finite logarithmic energy.

Notice here that if we change the variable of integration to $x\to cx+b$ and $y\to cy+b$, where $c\neq 0$, then, with

$$\mu_{c,b}=((\cdot-b)/c)_{\#}\mu,$$

($\cdot/c$ standing for the multiplication by $1/c$), and for a given function ϕ, the push forward $\phi_{\#}\mu$ is the measure defined by $\phi_{\#}\mu(A) = \mu(\{x : \phi(x) \in A\})$ for any Borel measurable A. Therefore we have

$$\begin{aligned} I_{\mathrm{V}}(\mu) &= \int V(cx+b)\mu_{b,c}(\mathrm{d}x) - \iint \log|cx-cy|\mu_{b,c}(\mathrm{d}x)\mu_{b,c}(\mathrm{d}y) \\ &= I_{\mathrm{V}(\cdot c+b)-\log(c)}(\mu_{b,c}) = I_{\mathrm{V}(\cdot c+b)}(\mu_{b,c}) - \log c \end{aligned} \tag{32}$$

which in turn results with

$$I_{\mathrm{V}} = I_{\mathrm{V}(\cdot c+b)-\log(c)} = I_{\mathrm{V}(\cdot c+b)} - \log(c).$$

3.3. Logarithmic Potentials and Chebyshev Polynomials

Recall that the *Chebyshev polynomials of the first kind* $T_n(x)$ are defined by

$$T_n(\cos\theta) = \cos(n\theta) \tag{33}$$

see for example, [32]. Alternatively, they are given by the recursion relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad T_0(x) = 1,\ T_1(x) = x.$$

T_n are the orthogonal polynomials for the *arcsine law* $\mathbb{1}_{[-1,1]}(x)\frac{1}{\pi\sqrt{1-x^2}}$. The following lemma is due to Haagerup [22] and we reproduce the proof here for completeness.

Lemma 3.1 (Haagerup).

(a) *For any real* $x, y \in [-2,2], x \neq y$, *we have*

$$\log|x-y| = -\sum_{n=1}^{\infty} \frac{2}{n} T_n\left(\frac{x}{2}\right) T_n\left(\frac{y}{2}\right)$$

where the series here is convergent on $x \neq y$.

(b) *If* $x > 2$ *and* $y \in [-2,2]$, *we have*

$$\log|x-y| = \log\left|\frac{x+\sqrt{x^2-4}}{2}\right| - \sum_{n=1}^{\infty}\frac{2}{n}\left(\frac{x-\sqrt{x^2-4}}{2}\right)^n T_n\left(\frac{y}{2}\right)$$

where the series is absolutely convergent.

(c) *The logarithmic potential of a measure on* $[-2,2]$ *is given by*

$$\int \log|x-y|\mu(\mathrm{d}x) = -\sum \frac{2}{n} T_n\left(\frac{x}{2}\right)\int T_n\left(\frac{y}{2}\right)\mu(\mathrm{d}y) \tag{34}$$

where this series makes sense pointwise.

(d) *The logarithmic energy of the measure* μ *is given by*

$$\iint \log|x-y|\mu(\mathrm{d}x)\mu(\mathrm{d}y) = -\sum_{n=1}^{\infty}\frac{2}{n}\left(\int T_n\left(\frac{x}{2}\right)\mu(\mathrm{d}x)\right)^2. \tag{35}$$

In particular $\iint \log|x-y|\mu(\mathrm{d}x)\mu(\mathrm{d}y)$ *is finite if and only if* $\sum_{n=1}^{\infty}\frac{2}{n}\left(\int T_n\left(\frac{x}{2}\right)\mu(\mathrm{d}x)\right)^2$ *is finite.*

Proof. We first point out that for any complex number $z \neq 1$, with $|z| = 1$, one has that

$$\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \tag{36}$$

where we take the branch of log on $\mathbb{C}\backslash(-\infty, 0]$. Now, write $x = 2\cos u$ and $y = 2\cos v$, and observe

$$x - y = 2(\cos u - \cos v) = 4\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{v-u}{2}\right),$$

and hence,

$$\begin{aligned}
\log|x-y| &= \log\left|2\sin\left(\frac{u+v}{2}\right)\right| + \log\left|2\sin\left(\frac{v-u}{2}\right)\right| \\
&= \log|1 - \mathrm{e}^{\mathrm{i}(u+v)}| + \log|1 - \mathrm{e}^{\mathrm{i}(v-u)}| \\
&= \mathrm{Re}\left(\log(1 - \mathrm{e}^{\mathrm{i}(u+v)}) + \log(1 - \mathrm{e}^{\mathrm{i}(v-u)})\right) \\
&= -\sum_{n=1}^{\infty} \frac{1}{n}\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}n(u+v)} + \mathrm{e}^{\mathrm{i}n(v-u)}\right) \\
&= -\sum_{n=1}^{\infty} \frac{1}{n}\left(\cos(n(u+v)) + \cos(n(v-u))\right) \\
&= -\sum_{n=1}^{\infty} \frac{2}{n}\cos(nu)\cos(nv) \\
&= -\sum_{n=1}^{\infty} \frac{2}{n} T_n\left(\frac{x}{2}\right) T_n\left(\frac{y}{2}\right).
\end{aligned}$$

For the case $x > 2$ and $|y| \leq 2$, then write $x = 2\cosh u = \mathrm{e}^u + \mathrm{e}^{-u}$, where $u = \log\frac{x+\sqrt{x^2-4}}{2}$ and $y = 2\cos v$, thus

$$\begin{aligned}
\log|x-y| &= \log(\mathrm{e}^u(1 - \mathrm{e}^{-u+\mathrm{i}v})(1 - \mathrm{e}^{-u-\mathrm{i}v})) \\
&= u + \log(1 - \mathrm{e}^{-u+\mathrm{i}v}) + \log(1 - \mathrm{e}^{-u-\mathrm{i}v}) \\
&= u - \sum_{n=1}^{\infty} \frac{2}{n}\mathrm{e}^{-nu}\cos(nv).
\end{aligned}$$

For the second part, for given $-1 < r < 1$ we introduce the kernel

$$L_r(x, y) := -\sum_{n\geq 1} \frac{2r^n}{n} T_n\left(\frac{x}{2}\right) T_n\left(\frac{y}{2}\right).$$

This can be computed for $x = 2\cos u$ and $y = 2\cos v$ for $u, v \in [0, \pi)$ with $u \neq v$ as

$$\begin{aligned}
&L_r(2\cos u, 2\cos v) \\
&\quad = -\sum_{n\geq 1} \frac{2r^n}{n}\cos(nu)\cos(nv)
\end{aligned}$$

$$= -\sum_{n=1}^{\infty} \frac{r^n}{n} \left(\cos(n(u+v)) + \cos(n(u-v))\right)$$
$$= -\sum_{n=1}^{\infty} \frac{r^n}{n} \mathrm{Re}\left(\mathrm{e}^{in(u+v)} + \mathrm{e}^{in(u-v)}\right)$$
$$=_{(36)} \log|1-\mathrm{e}^{\mathrm{i}(u+v)}| + \log|1-\mathrm{e}^{\mathrm{i}(u-v)}|$$
$$= \frac{1}{2}\left(\log(1+r^2-2r\cos(u+v)) + \log(1+r^2-2r\cos(u-v))\right)$$

Next, for any θ,

$$4 \ge 1 + r^2 - 2r\cos\theta \ge \left(\frac{1+r}{2}\right)^2 (2-2\cos\theta)$$

which results with

$$\log 4 \ge L_r(2\cos u, 2\cos v) \ge 2\log\frac{1+r}{2} + \log|2\cos u - 2\cos v|,$$

or for $x, y \in [-2,2], x \neq y$,

$$\log 4 \ge L_r(x,y) \ge 2\log\frac{1+r}{2} + \log|x-y|.$$

This combined with Fatou's lemma yields that

$$\lim_{r\to 1^-} \int L_r(x,y)\mu(\mathrm{d}y) = \int \log|x-y|\mu(\mathrm{d}y).$$

The rest follows. □

The first consequence of the above proposition is the computation of the well-known arcsine law of an interval; [37].

Corollary 3.2. *If* $\omega(dx) = \mathbb{1}_{[-2,2]}(x)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}}$ *is the arcsine law of the interval* $[-2,2]$*, then*

$$\int \log|x-y|\omega(\mathrm{d}y) = \begin{cases} 0, & |x| \le 2 \\ \log\frac{|x|+\sqrt{x^2-4}}{2}, & |x| > 2. \end{cases} \tag{37}$$

If μ *is a signed measure on* $[-2,2]$ *with finite total variation and finite logarithmic energy, then*

$$\int \log|x-y|\mu(\mathrm{d}y) = c \text{ almost everywhere for all } x \in [-2,2] \tag{38}$$

if and only if $\mu(\mathrm{d}x) = \mathbb{1}_{[-2,2]}(x)\frac{\mu([-2,2])\ \mathrm{d}x}{\pi\sqrt{4-x^2}}$*. Here, almost everywhere is understood with respect to the Lebesgue measure. Additionally, the constant* c *must be* 0.

Proof. Because the density of ω is even, it suffices to prove (37) for $x > 2$ or $x \in [-2,2]$. Equation (37) follows from the lemma and the fact that the series in (34) is convergent and is convergent also in $L^2(\mathbb{1}_{[-2,2]}(x)\frac{1}{\pi\sqrt{4-x^2}})$.

For the second part, integrating (38) with respect to the arcsine law and exchanging the integration one obtains that $c = 0$. Now, using equality (34) we obtain that $\int T_n(\frac{x}{2})\mu(\mathrm{d}y) = 0$ for all $n \geq 1$ and thus μ and $\mathbb{1}_{[-2,2]}(x)\frac{\mu([-2,2])\ \mathrm{d}x}{\pi\sqrt{4-x^2}}$ have the same moments and consequently must be equal. $\square$

3.4. A Connection with Fourier Analysis

Take the map

$$\Theta : [0, \pi] \ni \theta \to 2\cos\theta \in [-2, 2].$$

For a given measure μ on $[-2, 2]$ we define $\tilde{\mu} = \mu \circ \Theta$, the measure on $[0, \pi]$ such that

$$\tilde{\mu}(A) = \mu(\Theta(A))$$

for every measurable set A in $[0, \pi]$. The map $\mu \to \tilde{\mu}$ from measures on $[-2, 2]$ into the set of measures on $[0, \pi]$ is a one to one and onto. The advantage of using this comes from

$$\alpha_n := \int T_n\left(\frac{x}{2}\right)\mu(\mathrm{d}x) = \int \cos(n\theta)\tilde{\mu}(\mathrm{d}\theta), \quad n \geq 0, \tag{39}$$

which tells us that the "moments" of μ with respect to Chebyshev's polynomials are seen as the Fourier coefficients of a measure on $[0, \pi]$.

The following result is standard and we state it without proof.

Proposition 3.3. *Given a sequence $\{\alpha_n\}_{n\geq 0}$, with $\alpha_0 = 1$, the following are equivalent:*

1. *There exists a measure on $[-2, 2]$ such that $\alpha_n = \int T(\frac{x}{2})\mu(\mathrm{d}x)$;*
2. *$\{\alpha_n\}_{n\geq 0}$ is a bounded sequence and*

$$\langle \mu, \phi \rangle := \int_{-2}^{2} \frac{\phi(x)}{\pi\sqrt{4-x^2}}\, \mathrm{d}x + 2\sum_{n=1}^{\infty} \alpha_n \int_{-2}^{2} T_n\left(\frac{x}{2}\right)\frac{\phi(x)}{\pi\sqrt{4-x^2}}\, \mathrm{d}x$$

defines a nonnegative distribution, i.e., for any smooth nonnegative function $\phi : [-2, 2] \to \mathbb{R}_+$,

$$\langle \mu, \phi \rangle \geq 0.$$

In particular, if $\sum_{n=1}^{\infty} |\alpha_n|$ is convergent, then there is a measure μ on $[-2, 2]$ such that $\alpha_n = \int T(\frac{x}{2})\mu(\mathrm{d}x)$, if and only if

$$u(x) := 1 + 2\sum_{n=1}^{\infty} \alpha_n T_n\left(\frac{x}{2}\right) \geq 0 \quad \text{for all } x \in [-2, 2]. \tag{40}$$

In this case the measure μ is given by $\mu(\mathrm{d}x) = \frac{u(x)\mathrm{d}x}{\pi\sqrt{4-x^2}}$.

4. The Planar Limit of Analytic Matrix Models, The Main Results

Given a continuous function f on $[-2, 2]$, we define

$$\beta_n(f) = \int_{-2}^{2} f(x)T_n\left(\frac{x}{2}\right)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}}, \quad n \geq 0. \tag{41}$$

Notice that if f is bounded by a C^{k-1} function and piecewise C^k for some $k \geq 0$, using the Fourier interpretation and repeated integrations by parts we learn that $\beta_n(f) = o(n^{-k})$. Next define the orthogonal polynomials

$$\tilde{T}_n(x) = \sqrt{2}T_n(x/2) \tag{42}$$

for $n \geq 1$ and $\tilde{T}_0 = T_0 = 1$. These provide a Hilbert basis of $L^2(\mathbb{1}_{[-2,2]}(x)\frac{1}{\pi\sqrt{4-x^2}}\mathrm{d}x)$.

Theorem 4.1. *Assume that V is a C^2 and piecewise C^3 function on $[-2,2]$ and $A \in \mathbb{R}$ a constant. Then, there is a unique signed measure μ on $[-2,2]$ of finite total variation which solves*

$$\begin{cases} 2\int \log|x-y|\mu(\mathrm{d}x) = V(x) + C \text{ almost everywhere for } x \in [-2,2], \\ \mu([-2,2]) = A. \end{cases}$$

where almost everywhere is with respect to the Lebesgue measure on $[-2,2]$. The solution μ is given by $\mu(\mathrm{d}x) = \frac{u(x)\ \mathrm{d}x}{\pi\sqrt{4-x^2}}$ where

$$u(x) = A - \frac{1}{2}\int_{-2}^{2} \frac{yV'(y)\mathrm{d}y}{\pi\sqrt{4-y^2}} - \frac{x}{2}\int_{-2}^{2} \frac{V'(y)\ \mathrm{d}y}{\pi\sqrt{4-y^2}} + \frac{4-x^2}{2}\int_{-2}^{2} \frac{V'(x)-V'(y)}{x-y}\frac{\mathrm{d}y}{\pi\sqrt{4-y^2}}. \tag{43}$$

In addition, the constant C must be given by $C = -\int_{-2}^{2} \frac{V(x)\mathrm{d}x}{\pi\sqrt{4-x^2}}$.

Proof. In the first place, the uniqueness is clear because of Corollary 3.2.

To prove the rest we first write the function V

$$V(x) = \sum_{n=0}^{\infty} \langle \tilde{T}_n, V\rangle \tilde{T}_n(x) = \beta_0(V) + 2\sum_{n=1}^{\infty} \beta_n(V) T_n\left(\frac{x}{2}\right)$$

where the inner product is taken in $L^2(\mathbb{1}_{[-2,2]}(x)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}})$ and point out that the regularity of V implies that $\beta_n(V) = O(1/n^2)$. Invoking representation (34), results with

$$-2\sum_{n\geq 1}\frac{2}{n}\left(\int T_n\left(\frac{y}{2}\right)\mu(\mathrm{d}y)\right)T_n\left(\frac{x}{2}\right) = C + \beta_0(V) + 2\sum_{n=1}^{\infty}\beta_n(V)T_n\left(\frac{x}{2}\right).$$

Thus, equating the coefficients, we must have now $C = -\beta_0(V)$ and

$$\int T_n\left(\frac{x}{2}\right)\mu(\mathrm{d}x) = -\frac{n}{2}\beta_n(V)$$

which, means that $\mu(\mathrm{d}x) = \frac{u(x)\mathrm{d}x}{\pi\sqrt{4-x^2}}$, for

$$u(x) = A - \sum_{n=1}^{\infty} n\beta_n(V)T_n\left(\frac{x}{2}\right).$$

Here is the point where we need the C^3 assumption to make sure this series converges absolutely since in this case $n\beta_n(V) = o(1/n^2)$.

To prove equality (43), our task now is to show that

$$
\begin{aligned}
-\sum_{n=1}^{\infty} n\beta_n(V)T_n\left(\frac{x}{2}\right)
&= -\frac{1}{2}\int_{-2}^{2} \frac{yV'(y)\mathrm{d}y}{\pi\sqrt{4-y^2}} - \frac{x}{2}\int_{-2}^{2} \frac{V'(y)\mathrm{d}y}{\pi\sqrt{4-y^2}} \\
&\quad + \frac{4-x^2}{2\pi^2}\int_{-2}^{2} \frac{V'(y)-V'(x)}{x-y}\frac{\mathrm{d}y}{\sqrt{4-y^2}}.
\end{aligned}
$$

Notice that both sides of this equation are linear functions of V and thus by a simple approximation argument it suffices to check it for the case of $V(x) = T_m(\frac{x}{2})$ for some $m \geq 1$. After making the change of variables $x = 2\cos u, y = 2\cos v$, this identity reduces to checking that

$$
\begin{aligned}
-\cos(nu) &= -\frac{1}{\pi}\int_{0}^{\pi} \frac{\cos v \sin nv}{\sin v}\,\mathrm{d}v - \frac{\cos u}{\pi}\int_{0}^{\pi} \frac{\sin nv}{\sin v}\,\mathrm{d}v \\
&\quad + \frac{\sin u}{\pi}\int_{0}^{\pi} \frac{\sin(nu)\sin v - \sin(nv)\sin u}{(\cos u - \cos v)\sin v}\,\mathrm{d}v.
\end{aligned}
$$

Now, instead of checking this we look at the generating functions of the right and left hand sides. Specifically, using the fact that for $-1 < r < 1$ and $t \in [0, \pi]$,

$$
\sum_{n=1}^{\infty} r^{n-1}\cos(nt) = \frac{\cos t - r}{1 - 2r\cos t + r^2}
$$

and

$$
\sum_{n=1}^{\infty} r^{n-1}\sin(nt) = \frac{\sin t}{1 - 2r\cos t + r^2}, \tag{44}
$$

the rest follows from straightforward computations. □

Proposition 4.1. *If $\mu \in \mathcal{P}([-2,2])$ and V is a C^2 and piecewise C^3 function on $[-2,2]$, then*

$$
I_{\mathrm{V}}(\mu) = \beta_0(V) + 2\sum_{n=1}^{\infty}\left(\beta_n(V)\alpha_n + \frac{\alpha_n^2}{n}\right) \tag{45}
$$

where

$$
\alpha_n = \int T_n\left(\frac{x}{2}\right)\mu(\mathrm{d}x). \tag{46}
$$

Furthermore, we also have that

$$I_{\mathrm{V}}(\mu) \geq \beta_0(V) - \frac{1}{2}\sum_{n=1}^{\infty} n\beta_n(V)^2 \tag{47}$$

with equality if and only if $\alpha_n = -\beta_n(V)/2$ *and*

$$1 - \sum_{n=1}^{\infty} n\beta_n(V)T_n\left(\frac{x}{2}\right) \geq 0 \quad \textit{for any } x \in [-2,2]. \tag{48}$$

In this case,

$$\mu(\mathrm{d}x) = \left(1 - \sum_{n=1}^{\infty} n\beta_n(V)T_n\left(\frac{x}{2}\right)\right)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}}. \tag{49}$$

Proof. One can write

$$\int V\ \mathrm{d}\mu = \beta_0(V) + 2\sum_{n=1}^{\infty}\beta_n(V)\int T_n\left(\frac{x}{2}\right)\mu(\mathrm{d}x) = \beta_0 + 2\sum_{n=1}^{\infty}\beta_n(V)\alpha_n.$$

To prove (47), one needs to complete the square in (45) to get that

$$I_{\mathrm{V}}(\mu) = \beta_0(V) - \frac{1}{2}\sum_{n=1}^{\infty} n\beta_n(V)^2 + \sum_{n=1}^{\infty}\frac{2}{n}\left(\alpha_n + \frac{n\beta_n(V)}{2}\right)^2.$$

This implies inequality (47). The equality is attained only for the case $\alpha_n = -\frac{n\beta_n(V)}{2}$ which, cf. (40) determines a measure on $[-2,2]$ if and only if (48) is satisfied. The rest follows easily. □

We arrive at a necessary and sufficient condition for deciding that an equilibrium measure on $[-2,2]$ has full support.

Corollary 4.2. *Assume that V is a C^2 and piecewise C^3 function on $[-2,2]$. Then, the equilibrium measure on $[-2,2]$ has full support if and only if*

$$1 - \sum_{n=1}^{\infty} n\beta_n(V)T_n\left(\frac{x}{2}\right) > 0 \quad \textit{for } x \textit{ on a dense subset of } [-2,2]. \tag{50}$$

In addition, in this case, we also have that

$$\inf_{\mu\in\mathcal{P}([-2,2])} I_{\mathrm{V}}(\mu) = \beta_0(V) - \frac{1}{2}\sum_{n=1}^{\infty} n\beta_n(V)^2.$$

Proof. Condition (50) and the previous Proposition guarantee that there is a measure μ with full support such that $\int_{-2}^{2} T_n(\frac{x}{2})\mu(\mathrm{d}x) = -\frac{n\beta_n}{2}$.

The other way around works as follows. Assume that μ_{V} is the equilibrium measure on $[-2,2]$ and has full support. What we need to show is that (50) is satisfied.

Let $\alpha_n = \int_{-2}^{2} T_n(\frac{x}{2})\mu_{\rm V}(\mathrm{d}x)$. Then, for any other measure ν with $I_{\rm V}(\nu) < \infty$ on $[-2, 2]$, from (45) we obtain that

$$
\begin{aligned}
&I_{\rm V}((1-\epsilon)\mu_{\rm V} + \epsilon\nu) \\
&\quad = \beta_0(V) - \frac{1}{2}\sum_{n=1}^{\infty} n\beta_n(V)^2 + \sum_{n=1}^{\infty}\frac{2}{n}\left((1-\epsilon)\alpha_n + \epsilon\alpha_n' + \frac{n\beta_n(V)}{2}\right)^2
\end{aligned}
$$

where $\alpha_n' = \int T_n(\frac{x}{2})\nu(\mathrm{d}x)$. Since $I_{\rm V}(\mu_{\rm V})$ is the minimum of $I_{\rm V}(\nu)$ over all probability measures on $[-2, 2]$, differentiation with respect to $\epsilon > 0$ at 0 yields

$$\sum_{n\geq 1}\frac{1}{n}\left(\alpha_n + \frac{n\beta_n(V)}{2}\right)(\alpha_n' - \alpha_n) \geq 0. \tag{51}$$

Now we consider measures of the form

$$\nu(\mathrm{d}x) = (1 + \delta\phi(x))\mu_{\rm V}(\mathrm{d}x)$$

where ϕ is a polynomial such that $\int \phi \, \mathrm{d}\mu = 0$ and δ is small in absolute value. Applying this for $\pm\delta$, in (51), where δ is small enough, we obtain that

$$\sum_{n\geq 1}\frac{1}{n}\left(\alpha_n + \frac{n\beta_n(V)}{2}\right)\int T_n\left(\frac{x}{2}\right)\phi(x)\mu_{\rm V}(\mathrm{d}x) = 0. \tag{52}$$

A word of caution is in order here. We need to justify that the measure ν has finite logarithmic energy, namely that

$$\sum_{n\geq 1}\frac{1}{n}\left(\int T_n\left(\frac{x}{2}\right)(1 + \delta\phi(x))\mu_{\rm V}(\mathrm{d}x)\right)^2 < \infty.$$

This actually follows easily for each polynomial $\phi = T_k$ for some $k \geq 0$ from the fact that $2T_kT_l = T_{|k-l|} + T_{k+l}$ for any $k, l \geq 0$.

Because of (31) we have that $2\int \log|x - y|\mu_{\rm V}(\mathrm{d}y) = V(x) + C$, almost surely (with respect to the Lebesgue measure) on $[-2, 2]$, and from Theorem 4.1, the density $g(x)$ of $\mu_{\rm V}$ is given by

$$g(x) = A_1\sqrt{4-x^2}\int_{-2}^{2}\frac{V'(y) - V'(x)}{\sqrt{4-y^2}(y-x)}\,\mathrm{d}y + \frac{A_2 + A_3x}{\sqrt{4-x^2}}$$

for some constants A_1 and A_2. In particular, since V is C^3 it implies that $g(x) = \frac{h(x)}{\pi\sqrt{4-x^2}}$ for some continuous function h. Since the measure $\mu_{\rm V}$ has full support, $h(x) > 0$ on a dense set.

Next, we observe that because $\beta_n(V)$ is square summable in $L^2(\mathbb{1}_{[-2,2]}(x)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}})$,

$$R(x) := \sum_{n\geq 1}\frac{1}{n}\left(\alpha_n + \frac{n\beta_n(V)}{2}\right)T_n\left(\frac{x}{2}\right)$$

is convergent in $L^2(\mathbb{1}_{[-2,2]}(x)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}})$, therefore we deduce from (52) that

$$\int_{-2}^{2} R(x)\phi(x)h(x)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = 0,$$

for any polynomial ϕ. This easily implies that $R(x) = 0$ almost everywhere and this in turn results with $\alpha_n = -n\beta_n(V)/2$. The rest follows. □

Theorem 4.2. *If the equilibrium measure of a V which is C^2 and piecewise C^3 on $[-2,2]$ has full support then,*

$$\begin{aligned} I_{\mathrm{V}} &= \inf_{\mu\in\mathcal{P}([-2,2])} I_{\mathrm{V}}(\mu) \\ &= \int_{-2}^{2} \frac{V(x)\mathrm{d}x}{\pi\sqrt{4-x^2}} + \int_0^1 t\left[\left(\int_{-2}^{2}\frac{xV'(tx)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)^2 + \left(\int_{-2}^{2}\frac{V'(tx)\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)^2\right]\mathrm{d}t \\ &= V(0) - \int_0^1 \frac{1}{t}\left[-1 + \left(1 - \int_{-2}^{2}\frac{txV'(tx)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)^2 + \left(\int_{-2}^{2}\frac{tV'(tx)\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)^2\right]\mathrm{d}t. \end{aligned}$$

Proof. According to Corollary 4.2, we have

$$I_{\mathrm{V}} = \beta_0(V) - \frac{1}{2}\sum_{n\geq 1} n\beta_n(V)^2.$$

Thus our task is to prove that

$$\frac{1}{2}\sum_{n\geq 1} n\beta_n(V)^2 = \int_0^1 t\left[\left(\int_{-2}^{2}\frac{xV'(tx)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)^2 + \left(\int_{-2}^{2}\frac{V'(tx)\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)^2\right]\mathrm{d}t.$$

A polarization argument shows that this is equivalent to proving that for any C^3 potentials V_1 and V_2

$$\begin{aligned} \frac{1}{2}\sum_{n\geq 1} n\beta_n(V_1)\beta_n(V_2) &= \int_0^1 t\left(\int_{-2}^{2}\frac{xV_1'(tx)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)\left(\int_{-2}^{2}\frac{xV_2'(tx)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)\mathrm{d}t \\ &\quad + \int_0^1 t\left(\int_{-2}^{2}\frac{V_1'(tx)\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)\left(\int_{-2}^{2}\frac{V_2'(tx)\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)\mathrm{d}t. \end{aligned} \tag{53}$$

To do this, because of the linearity in V_1 and V_2 and the fact that polynomials are dense (with respect to C^3 topology) in the set of smooth functions on $[-2,2]$, it suffices to check this for $V_1(x) = x^k$ and $V_2(x) = x^m$. If k or m is zero, both sides of (53) are zero, therefore we need to check this for $k, m \geq 1$.

Now, we use

$$x^{2n} = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n}{k} T_{2n-2k}(x) + \frac{1}{2^{2n}}\binom{2n}{n},$$

and

$$x^{2n+1} = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} T_{2n+1-2k}(x)$$

from which a direct calculation yields

$$\int_{-2}^{2} x^i T_n\left(\frac{x}{2}\right) \frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = \binom{i}{\frac{i-n}{2}}, \tag{54}$$

with the convention that $\binom{i}{p+1/2} = 0$ for $p \in \mathbb{Z}$ and $\binom{i}{p} = 0$ for $p < 0$. Therefore, (53) becomes in this case

$$\frac{1}{2}\sum_{n\geq 1} n \binom{m}{\frac{m-n}{2}} \binom{k}{\frac{k-n}{2}} = \frac{mk}{4(m+k)} \binom{m}{\frac{m}{2}} \binom{k}{\frac{k}{2}} + \frac{mk}{(m+k)} \binom{m-1}{\frac{m-1}{2}} \binom{k-1}{\frac{k-1}{2}}.$$

In the case m, k have different parities, then the above expression is 0. If they have the same parities, the equality follows from the next lemma. □

Lemma 4.3. *The following identities hold*

$$\sum_p p \binom{2l_1}{l_1 - p} \binom{2l_2}{l_2 - p} = \frac{l_1 l_2}{2(l_1 + l_2)} \binom{2l_1}{l_1} \binom{2l_2}{l_2}$$

$$\sum_p (2p+1) \binom{2l_1 + 1}{l_1 - p} \binom{2l_2 + 1}{l_2 - p} = \frac{(2l_1+1)(2l_2+1)}{l_1 + l_2 + 1} \binom{2l_1}{l_1} \binom{2l_2}{l_2}$$

with the convention that $\binom{j}{q} = 0$ *for* $q < 0$ *or* $q > j$.

Proof. These identities can be checked with the *zb* package written for Mathematica. For details on this we refer the reader to the wonderful book [35]. For completeness we give here the main calculation.

The first identity is equivalent to

$$h(l_1, l_2) := \sum_p \frac{2p(l_1 + l_2)\binom{2l_1}{l_1 - p}\binom{2l_2}{l_2 - p}}{l_1 l_2 \binom{2l_1}{l_1}\binom{2l_2}{l_2}} = 1.$$

Let us denote

$$f(l_1, l_2, p) = \frac{2p(l_1 + l_2)\binom{2l_1}{l_1 - p}\binom{2l_2}{l_2 - p}}{l_1 l_2 \binom{2l_1}{l_1}\binom{2l_2}{l_2}}.$$

The idea of the *zb* method for our case is to write

$$f(l_1 + 1, l_2, p) - f(l_1, l_2, p) = g(l_1, l_2, p+1) - g(l_1, l_2, p). \tag{55}$$

and this proves that for all $l_1 \geq 1$ one has $h(l_1, l_2) = h(1, l_2)$. Since it is is immediate to show that $h(1, l_2) = 1$, the rest follows as soon as we know that $g(l_1, l_2, p) = 0$ for $p = 1$ and for large p.

The whole point of the zb method is to actually compute the function $g(l_1, l_2, p)$. We will refer the reader for the details to [35] and will give here just the results obtained with Mathematica.

$$g(l_1, l_2, p) = -\frac{2p(p-1)\binom{2l_1+1}{l_1+p}\binom{2l_2-1}{l_2-p}}{l_1(2l_1+1)\binom{2l_1}{l_1}\binom{2l_2}{\lambda_2}}.$$

Notice that for $p \geq \min\{l_1+2, l_2+1\}, g(l_1, l_2, p) = 0$. One can directly check (55), by dividing both sides by $f(l_1, l_2, p)$ which reduces it to an identity in the field $\mathbb{Q}(l_1, l_2, p)$.

For the second identity, as in the preceding argument, we want to show that

$$h(l_1, l_2) := \sum_p \frac{(2p+1)(l_1+l_2+1)\binom{2l_1+1}{l_1-p}\binom{2l_2+1}{l_2-p}}{(2l_1+1)(2l_2+1)\binom{2l_1}{l_1}\binom{2l_2}{l_2}} = 1.$$

Defining

$$f(l_1, l_2, p) = \frac{(2p+1)(l_1+l_2+1)\binom{2l_1+1}{l_1-p}\binom{2l_2+1}{l_2-p}}{(2l_1+1)(2l_2+1)\binom{2l_1}{l_1}\binom{2l_2}{l_2}},$$

the corresponding companion in this case is

$$g(l_1, l_2, p) = -\frac{p^2(l_2+1)^2\binom{2l_1+2}{l_1-p+1}\binom{2l_2}{l_2-p}}{\binom{2l_1+2}{l_1+1}\binom{2l_2}{l_2}}.$$

Equation (55) is satisfied and $g(l_1, l_2, 0) = 0$ and $g(l_1, l_2, p) = 0$ for $p \geq \min\{l_1+1, l_2+1\}$. This proves that $h(l_1, l_2) = h(0, l_2)$. Now, $h(0, l_2) = 1$ which ends the proof. □

Before we state the next result, for a C^3 potential V, we define

$$\psi_{b,c}(x) := \int_{-2}^{2} \frac{V'(cx+b) - V'(cy+b)}{x-y} \frac{\mathrm{d}y}{\pi\sqrt{4-y^2}}. \tag{56}$$

Theorem 4.3. *Assume V is an admissible potential on $\mathbb{R}$. Then the equilibrium measure on $\mathbb{R}$ associated to V has support the interval $[-2c+b, 2c+b]$ if and only if (c, b) is the unique absolute maximizer in $\mathbb{R}_+^* \times \mathbb{R}$ of*

$$H(c, b) := \log c - \frac{1}{2}\int_{-2}^{2} V(cx+b)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} \tag{57}$$

and

$$\psi_{b,c} > 0 \quad \textit{on a dense subset of } [-2, 2]. \tag{58}$$

If in addition V is a C^2 and piecewise C^3 potential on a neighborhood of the support $[-2c+b, 2c+b]$, then (b, c) is a solution of

$$\begin{cases} \int_{-2}^{2} cxV'(cx+b)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = 2 \\ \int_{-2}^{2} V'(cx+b)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = 0. \end{cases} \tag{59}$$

In this case the equilibrium measure $\mu_{\rm V}$ *is given by*

$$\mu_{\rm V}(dx) = \mathbb{1}_{[-2c+b,2c+b]}(x)\frac{\psi_{b,c}((x-b)/c)\sqrt{4c^2-(x-b)^2}}{2c\pi}\,\mathrm{d}x$$

and

$$\begin{aligned} I_{\rm V} &= -\log c + \int_{-2}^{2} \frac{V(cx+b)\mathrm{d}x}{\pi\sqrt{4-x^2}} \\ &\quad - \int_0^c s\left[\left(\int_{-2}^{2}\frac{xV'(sx+b)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)^2 + \left(\int_{-2}^{2}\frac{V'(sx+b)\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)^2\right]\mathrm{d}s \\ &= V(b) - \log c - \int_0^c \frac{1}{s} \\ &\quad \times\left[-1+\left(1-\int_{-2}^{2}\frac{sxV'(sx+b)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)^2 + \left(\int_{-2}^{2}\frac{sV'(sx+b)\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)^2\right]\mathrm{d}s. \end{aligned} \tag{60}$$

Proof. If the support of $\mu_{\rm V}$ is the interval $[-2c+b, 2c+b]$, we have to prove first that (c, b) is the unique absolute maximizer of H. The function H appears in the literature as the F-functional of Mhaskar and Saff (see for instance [37, page 194]) and for the sake of completeness we adapt the proof of this first part from there.

Define the arcsine law of the interval $[-2c+b, 2c+b]$ to be

$$\omega_{c,b}(\mathrm{d}x) = \mathbb{1}_{[-2c+b,2c+b]}(x)\frac{\mathrm{d}x}{\pi\sqrt{4c^2-(x-b)^2}}.$$

A simple rescaling of Eq. (37), shows that $\int \log|x-y|\omega_{c,b}(\mathrm{d}y) \geq \log(c)$ for all x, with equality only for $x \in [-2c+b, 2c+b]$.

Integrating Eq. (31) against the measure $\omega_{c',b'}$ yields that

$$\int V(x)\omega_{c',b'}(\mathrm{d}x) - 2\int\int \log|x-y|\omega_{c',b'}(\mathrm{d}x)\mu_{\rm V}(\mathrm{d}y) \geq C$$

and thus, after interchanging the integrations,

$$\int V(x)\omega_{c',b'}(\mathrm{d}x) - 2\log(c') \geq C.$$

Because (31) is equality quasi-everywhere on $[-2c+b, 2c+b]$, this implies that we have equality in the above inequality for $c' = c$ and $b' = b$. In fact, this is the only case of equality as otherwise

$$\begin{aligned} C &= \int V(x)\omega_{c',b'}(\mathrm{d}x) - 2\log(c') \\ &\geq \int V(x)\omega_{c',b'}(\mathrm{d}x) - 2\int\int \log|x-y|\omega_{c',b'}(\mathrm{d}x)\mu_{\rm V}(\mathrm{d}y) \geq C, \end{aligned}$$

hence we must have that μ_V almost surely, $\log(c') = \int \log|x-y|\omega_{c',b'}(dy)$, which according to (37) is possible if and only if $\omega_{c',b'}$ is actually equal to $\omega_{c,b}$, or $c' = c$ and $b' = b$.

From (57), upon differentiation with respect to c and b, we deduce that

$$\int_{-2}^{2} cxV'(cx+b)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = 2 \quad \text{and} \quad \int_{-2}^{2} V'(cx+b)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = 0 \tag{61}$$

which combined with (43) proves (58).

To prove the converse, notice that because (c,b) is a maximizer of H, we have (61). It is then clear that the μ_V solves Eq. (43). What we have to prove is that this measure satisfies condition (31). To this end, it is sufficient to prove that for any $b' \in \mathbb{R}$ and $c' > 0$ one has

$$\int \left(V(x) - 2\int \log|x-y|\mu_V(\mathrm{d}y)\right)\omega_{c',b'}(\mathrm{d}x) \geq C.$$

Switching the integration in the double integral, and performing some elementary calculations, this inequality becomes equivalent to

$$\int \frac{V(c'x+b')\,\mathrm{d}x}{\pi\sqrt{4-x^2}} - 2\int\int \log|x-y|\omega_{c',b'}(\mathrm{d}x)\mu_V(\mathrm{d}y) \geq C.$$

This inequality is equality for $c' = c$ and $b' = b$, and thus $C = -H(c,b)$. If c' and b' are arbitrary, the inequality is a consequence of the fact that the left hand side of this inequality is greater than or equal to $-H(c',b')$ which in turn is by the hypothesis $\geq -H(c,b)$.

Identity (60) follows from Theorem 4.2 applied to $\tilde{V}(x) = V(cx+b)$. $\square$

In the case of even potentials, we know that the equilibrium measure is symmetric and thus in the preceding result we can always assume that $b = 0$ and this deserves a special statement because of its simplicity.

Corollary 4.4. *If V is a C^2, piecewise C^3 and even satisfying (5), its equilibrium measure is supported on the interval $[-2c, 2c]$ if and only if $c > 0$ is the unique maximizer of*

$$H(c) = \log c - \int_{0}^{2} \frac{V(cx)\mathrm{d}x}{\pi\sqrt{4-x^2}}$$

and

$$\psi_c(x) := \int_{-2}^{2} \frac{V'(cx)-V'(cy)}{x-y}\frac{\mathrm{d}y}{\pi\sqrt{4-y^2}}$$

is positive on a dense set of $[-2,2]$. In particular c solves

$$\int_{-2}^{2} cxV'(cx)\frac{\mathrm{d}x}{\sqrt{4-x^2}} = 2. \tag{62}$$

In this case the planar limit is

$$I_{\mathrm{V}} = V(0) - \log c - \int_0^c \frac{1}{s}\left[-1 + \left(1 - \int_0^2 sxV'(sx)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)^2\right]\mathrm{d}s. \quad (63)$$

We point out here an interesting property, namely, that the solutions (c, b) of the system (59) are critical points of the functional I_{V} from (60).

Proposition 4.5. *Let V be a C^1 potential on $\mathbb{R}$ and consider*

$$I_{\mathrm{V}}(u,v) = -\log u + \int_{-2}^{2} \frac{V(ux+v)\mathrm{d}x}{\pi\sqrt{4-x^2}}$$
$$- \int_0^u s\left[\left(\int_{-2}^{2}\frac{xV'(sx+v)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)^2 + \left(\int_{-2}^{2}\frac{V'(sx+v)\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)^2\right]\mathrm{d}s.$$

If (c, b) satisfy

$$\int_{-2}^{2}\frac{V'(cx+b)\mathrm{d}x}{\pi\sqrt{4-x^2}} = 0,$$

then

$$\left.\frac{\partial I_{\mathrm{V}}}{\partial v}\right|_{(c,b)} = 0. \quad (64)$$

If (c, b) satisfy (59), then

$$\left.\frac{\partial I_{\mathrm{V}}}{\partial u}\right|_{(c,b)} = 0. \quad (65)$$

In particular the critical points of H from (57) are also critical points of I_{V}.

Proof. To see (64), after differentiating with respect to v, we need to show that

$$\left.\frac{\partial I_{\mathrm{V}}}{\partial v}\right|_{(c,b)} = -\int_{-2}^{2}\frac{V'(cx+b)\mathrm{d}x}{\pi\sqrt{4-x^2}}$$
$$+ \int_0^c\int_{-2}^{2}\int_{-2}^{2}\frac{s(xy+4)V'(sx+b)V''(sy+b)}{4\pi^2\sqrt{(4-x^2)(4-y^2)}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s = 0.$$

Now we present the following result.

Lemma 4.6. *If $U \in C^1([-2,2])$ or is a formal power series, then the following holds*

$$\int_0^1\int_{-2}^2\int_{-2}^2 \frac{s(xy+4)U(sx)U'(sy)}{4\pi^2\sqrt{(4-x^2)(4-y^2)}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s = \int_{-2}^2 \frac{U(x)}{\pi\sqrt{4-x^2}}\,\mathrm{d}x \int_0^1\int_{-2}^2 \frac{sU'(sy)}{\pi\sqrt{4-y^2}}\,\mathrm{d}y\,\mathrm{d}s. \tag{66}$$

In particular, if U satisfies,

$$\int_{-2}^2 \frac{U(x)}{\pi\sqrt{4-x^2}}\,\mathrm{d}x = 0,$$

then

$$\int_0^1\int_{-2}^2\int_{-2}^2 \frac{s(xy+4)U(sx)U'(sy)}{4\pi^2\sqrt{(4-x^2)(4-y^2)}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s = 0.$$

Proof. By polarization, it suffices to show that for any two C^1 potentials or formal power series, U_1 and U_2 on $[-2,2]$, we have that

$$\begin{aligned}\int_0^1\int_{-2}^2\int_{-2}^2 &\frac{s(xy+4)(U_1(sx)U_2'(sy)+U_2(sx)U_1'(sy)}{4\pi^2\sqrt{(4-x^2)(4-y^2)}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\ &= \int_{-2}^2 \frac{U_1(x)}{\pi\sqrt{4-x^2}}\mathrm{d}x \int_0^1\int_{-2}^2 \frac{sU_2'(sy)}{\pi\sqrt{4-y^2}}\mathrm{d}y\,\mathrm{d}s\\ &\quad+\int_{-2}^2 \frac{U_2(x)}{\pi\sqrt{4-x^2}}\mathrm{d}x \int_0^1\int_{-2}^2 \frac{sU_1'(sy)}{\pi\sqrt{4-y^2}}\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

It is clear now that it suffices to check this for $U_1(x) = x^n$ and $U_2(x) = x^m$, which, with the help of (54), becomes

$$\begin{aligned}&\frac{1}{n+m+1}\left[\frac{n}{4}\binom{n}{\frac{n}{2}}\binom{m+1}{\frac{m+1}{2}} + \frac{m}{4}\binom{m}{\frac{m}{2}}\binom{n+1}{\frac{n+1}{2}}\right.\\ &\qquad\left.+m\binom{n}{\frac{n}{2}}\binom{m-1}{\frac{m-1}{2}} + n\binom{m}{\frac{m}{2}}\binom{n-1}{\frac{n-1}{2}}\right]\\ &\quad= \frac{m}{m+1}\binom{n}{\frac{n}{2}}\binom{m-1}{\frac{m-1}{2}} + \frac{n}{n+1}\binom{m}{\frac{m}{2}}\binom{n-1}{\frac{n-1}{2}}.\end{aligned}$$

Here we use the convention that $\binom{a}{b} = 0$ if b is not a nonnegative integer. As long as n and m have the same parity, both sides of the above expression are 0. Also due to the symmetry in n and m, it suffices to check this for $n = 2k$ and $m = 2l+1$. In this case, it is easy to prove that both sides are equal to

$$\frac{2l+1}{2l+2}\binom{2k}{k}\binom{2l}{l}.$$

□

Taking $U(x) = V'(cx+b)$ in the lemma, after a simple change of variables, the rest follows.

Equation (65) is clear from the fact that

$$I_{\mathrm{V}}(u,v) = V(v) - \log u$$
$$-\int_0^u \frac{1}{s}\left[-1+\left(1-\int_{-2}^{2}\frac{sxV'(sx+v)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)^2+\left(\int_{-2}^{2}\frac{sV'(sx+v)\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)^2\right]\mathrm{d}s.$$

and thus the u-derivative vanishes under the condition of (59). □

5. Examples and Computations with Analytic Matrix Models

5.1. Cases of One-Cut Potentials

With the result from Theorem 4.3, it is instructive to recover the classical results (see [37] where weaker regularity conditions are required) which guarantee that there is a one interval support of the equilibrium measure.

Corollary 5.1. *Assume that a C^3 potential V satisfying (5) is either convex or even with $xV'(x)$ increasing on $[0,\infty)$. Then the equilibrium measure has one interval support and the maximizer is non-degenerate (i.e. the Hessian of H is invertible at the maximizer). In addition, the function $\psi_{c,b}$ is positive on $[-2,2]$.*

Proof. First, we need to check that the function $H(b,c)$ has a unique maxima.

In the case V is convex, we show that H is concave. Indeed, the hessian of H at (c,b) is

$$(\mathrm{Hess}H)(c,b) = \begin{bmatrix} -\frac{1}{c^2} - \int_{-2}^{2}\frac{x^2V''(cx+b)\mathrm{d}x}{2\pi\sqrt{4-x^2}} & -\int_{-2}^{2}\frac{xV''(cx+b)\mathrm{d}x}{2\pi\sqrt{4-x^2}} \\ -\int_{-2}^{2}\frac{xV''(cx+b)\mathrm{d}x}{2\pi\sqrt{4-x^2}} & -\int_{-2}^{2}\frac{V''(cx+b)\mathrm{d}x}{2\pi\sqrt{4-x^2}} \end{bmatrix}$$

and strict concavity is equivalent to

$$\frac{1}{c^2} + \int_{-2}^{2}\frac{x^2V''(cx+b)\mathrm{d}x}{2\pi\sqrt{4-x^2}} > 0 \quad \text{and}$$

$$\frac{1}{c^2}\int_{-2}^{2}\frac{V''(cx+b)\mathrm{d}x}{2\pi\sqrt{4-x^2}} + \int_{-2}^{2}\frac{V''(cx+b)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\int_{-2}^{2}\frac{x^2V''(cx+b)\mathrm{d}x}{2\pi\sqrt{4-x^2}}$$
$$-\left(\int_{-2}^{2}\frac{xV''(cx+b)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)^2 > 0.$$

The only way either of these fail is if $V''(cx+b)=0$ for all $x\in[-2,2]$, which implies $V(x)=Ax+B$ for some constants A,B and all $x\in[-2c+b,2c+b]$. This in turn results with $F(c',b')=\log c'-B$ for all $c'<c$ which contradicts the assumption that (c,b) is a maximizer of F.

On the other hand, one can easily check that H is concave on $(0,\infty)\times\mathbb{R}$. This combined with strict concavity near (c,b) implies that the maximizer is unique.

In the case V is even and $xV'(x)$ is increasing, we may assume that $b=0$ and thus the function H becomes a function of one variable with

$$H'(c)=\frac{1}{c}-\int_{-2}^{2}\frac{xV'(cx)\mathrm{d}x}{2\pi\sqrt{4-x^2}}=\frac{1}{c}\left(1-\int_{0}^{2}\frac{cxV'(cx)\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)$$

Now, since the function $xV'(x)$, is increasing, one can see that $cH'(c)$ is decreasing and thus there is only at most one critical point of H. On the other hand, one can check that there is a maximizer of $H(c)$, hence we deduce that there is such a unique maximizer.

In addition to this, the Hessian of $H(c,b)$ at the maximizer $(c,0)$ is

$$(\mathrm{Hess}H)(c,b)=\begin{bmatrix}-\frac{1}{c^2}-\int_{-2}^{2}\frac{x^2V''(cx)\mathrm{d}x}{2\pi\sqrt{4-x^2}} & 0\\ 0 & -\int_{-2}^{2}\frac{V''(cx)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\end{bmatrix}$$

Now, using the fact that $H'(c)=0$ and a simple integration by parts reveals that

$$\begin{aligned}1=\int_{-2}^{2}\frac{cxV'(cx)\mathrm{d}x}{2\pi\sqrt{4-x^2}}&=\int_{-2}^{2}\frac{c^2V''(cx)\sqrt{4-x^2}\mathrm{d}x}{2\pi}\\&=\int_{-2}^{2}\frac{4c^2V''(cx)\mathrm{d}x}{2\pi\sqrt{4-x^2}}-\int_{-2}^{2}\frac{c^2x^2V''(cx)\mathrm{d}x}{2\pi\sqrt{4-x^2}}\end{aligned}$$

which implies

$$4\int_{-2}^{2}\frac{V''(cx)\mathrm{d}x}{2\pi\sqrt{4-x^2}}=\frac{1}{c^2}+\int_{-2}^{2}\frac{x^2V''(cx)\mathrm{d}x}{2\pi\sqrt{4-x^2}}. \qquad (*)$$

Now, if we denote $g(x)=xV'(x)$, then g is an increasing function on $[0,\infty]$ and therefore $x^2V''(x)=xg'(x)-g(x)>-g'(x)$ for all $x>0$. In particular we obtain that $c^2x^2V''(cx)>-g(cx)$ for all $x\in[0,2]$. Furthermore, from the equation determining c, we get

$$1=\int_{0}^{2}\frac{g(cx)\mathrm{d}x}{\pi\sqrt{4-x^2}}$$

which in turn implies

$$\frac{1}{c^2} + \int_0^2 \frac{c^2x^2V''(cx)\mathrm{d}x}{\pi\sqrt{4-x^2}} > \frac{1}{c^2} - \frac{1}{c^2}\int_0^2 \frac{g(cx)\mathrm{d}x}{\pi\sqrt{4-x^2}} = 0$$

and this means that quantities in (*) are positive, thus the Hessian of F at $(c, 0)$ is non-degenerate.

Having checked the uniqueness of the maximizer, we need to check the other condition. In the case of convex potentials, the non-negativity of $\psi_{c,b}$ follows from the fact that $\frac{V'(cx+b)-V'(cy+b)}{x-y} \geq 0$ for all $x, y \in [-2, 2]$. Furthermore, $\psi_{c,b}(x) = 0$, enforces $V'(cx + b) = V'(cy + b)$ for all $y \in [-2, 2]$, which in turn yields $V(c \cdot +b)$ is constant on $[-2, 2]$, something which is contradicted by the assumption that (c, b) is a maximizer of $H(c, b)$. Hence we actually obtain the stronger conclusion, namely $\psi_{c,b}(x) > 0$ on $[-2, 2]$.

In the case V is even and $xV'(x)$ increases on $[0,\infty]$, one can show that ψ_c is an even function and with simple manipulations of integrals that

$$\psi_c(x) = \int_0^2 \frac{xV'(cx) - yV'(cy)}{x^2 - y^2}\frac{\mathrm{d}y}{\pi\sqrt{4-y^2}}$$

which makes clear that $\psi_c(x) > 0$ for all $x \in [-2, 2]$. □

5.2. Analytic Planar Limits of Various Even Potentials

In this section we explicitly compute the planar limit of some 1-cut potentials, illustrating the formulas of Sect. 4. A typical example is the case where V is a smooth potential, which is analytic near the support of the equilibrium measure.

The easiest to deal with is the case of even potentials because in this case we can invoke Corollary 4.4 and reduce the problem of determining the support of the equilibrium measure to the maximization of a function of a single variable. In this case the planar limit is actually a one variable function of the right endpoint $2c$ of the equilibrium measure.

Assume that V is an even potential such that it has a power series expansion valid on a neighborhood of the support:

$$V(x) = \sum_{n=1}^{\infty} a_{2n}\frac{x^{2n}}{2n}. \tag{67}$$

In this case, from Corollary 4.4 we learn that

$$H(c) = \log c - \frac{1}{2}\sum_{n=1}^{\infty}\frac{a_{2n}c^{2n}}{2n}\int_{-2}^{2}\frac{x^{2n}\mathrm{d}x}{\pi\sqrt{4-x^2}} = \log c - \frac{1}{2}\sum_{n=1}^{\infty}\frac{a_{2n}c^{2n}}{2n}\binom{2n}{n}$$

where in the last equality we used Eq. (54). The critical points of this function satisfy (62) which becomes

$$\sum_{n=1}^{\infty} a_{2n}c^{2n}\binom{2n}{n} = 2. \tag{68}$$

If c is the maximizer of F, then, again from Corollary 4.4 and (54), the planar limit is given by

$$I_{\mathrm{V}} = -\log c + \int_0^c \frac{1}{t}\left[-1 + \left(1 - \frac{1}{2}\sum_{n=1}^{\infty} a_{2n} t^{2n}\binom{2n}{n}\right)^2\right] \mathrm{d}t.$$

Example 5.2. For $V(x) = a_{2n}\frac{x^{2n}}{2n}$, with $a_{2n} > 0$, and $n \geq 1$, the support of the equilibrium measure is $[-2c, 2c]$, where

$$c = \left(\frac{a_{2n}}{2}\binom{2n}{n}\right)^{-\frac{1}{2n}}.$$

In this case, the equilibrium measure is

$$\mu_{\mathrm{V}}(\mathrm{d}x) = \mathbb{1}_{[-2c,2c]}(x)\frac{1}{2\pi c}\psi_c(x/c)\sqrt{4c^2 - x^2}\mathrm{d}x,$$
$$\psi_c(x) = a_{2n}c^{2n-1}\sum_{l=0}^{n-1}\binom{2l}{l}x^{2(n-l-1)}$$

and the planar limit is

$$I_{\mathrm{V}} = \frac{\log a_{2n}}{2n} + \frac{\log\left(\binom{2n}{n}/2\right)}{2n} + \frac{3}{4n}.$$

To see this, one has to realize that (68) becomes in this case

$$a_{2n}c^{2n}\binom{2n}{n} = 2$$

which has only one positive solution, and this is the maximizer of $H(c) = \log c - \frac{a_{2n}c^{2n}}{4n}\binom{2n}{n}$. The rest of the equalities are straightforward calculations.

It is worth pointing out that in this example the potential is convex and thus, the equilibrium measure must be supported on a single interval.

For $n = 1$, we recover the semicircular law.

Example 5.3. Assume $V(x) = a_{2n}\frac{x^{2n}}{2n} + a_{2m}\frac{x^{2m}}{2m}$ with $a_{2m} > 0$ and $1 \leq n \leq m$. In this case the equilibrium measure has a single interval support if and only if

$$a_{2n} \geq -C_{nm}a_{2m}^{m/n} \tag{69}$$

where

$$C_{nm} = K_{nm}\left(\frac{2}{\binom{2m}{m} - \binom{2n}{n}K_{nm}}\right)^{\frac{m-n}{n}} \quad \text{with } K_{nm} = \min_{t\in[0,4]}\frac{\sum_{l=0}^{m-1}\binom{2l}{l}t^{m-l-1}}{\sum_{l=0}^{n-1}\binom{2l}{l}t^{n-l-1}}. \tag{70}$$

In this case, the support of μ_{V} is $[-2c, 2c]$ where c is the unique positive solution to

$$a_{2n}c^{2n}\binom{2n}{n} + a_{2m}c^{2m}\binom{2m}{m} = 2, \tag{71}$$

the equilibrium measure is

$$\mu_{\mathrm{V}}(\mathrm{d}x) = \frac{1}{2\pi c}\mathbb{1}_{[-2c,2c]}(x)\psi_c(x/c)\sqrt{4c^2-x^2}\,\mathrm{d}x$$
$$\psi_c(x) = a_{2n}c^{2n-1}\sum_{l=0}^{n-1}\binom{2l}{l}x^{2(n-l-1)} + a_{2m}c^{2m-1}\sum_{l=0}^{m-1}\binom{2l}{l}x^{2(m-l-1)}$$

and the planar limit is

$$I_{\mathrm{V}} = -\log c + \frac{c^{2n}a_{2n}}{2n}\binom{2n}{n} + \frac{c^{2m}a_{2m}}{2m}\binom{2m}{m} - \frac{c^{2(n+m)}a_{2n}a_{2m}}{4(n+m)}\binom{2n}{n}\binom{2m}{m} - \frac{c^{4n}a_{2n}^2}{16n}\binom{2n}{n}^2 - \frac{c^{4m}a_{2m}^2}{16m}\binom{2m}{m}^2.$$

To prove these, we need to look at the critical Eq. (68) and notice that for

$$f(c) = a_{2n}c^{2n}\binom{2n}{n} + a_{2m}c^{2m}\binom{2m}{m} - 2,$$

one has

$$f'(c) = 2c^{2n-1}\left(na_{2n}\binom{2n}{n} + ma_{2m}c^{2(m-n)}\binom{2m}{m}\right).$$

It is clear that f' has at most one positive root. If $c_0 > 0$ is the positive root of f', then $f'(c) < 0$ for $0 < c < c_0$ and $f'(c) > 0$ for $c > c_0$. If f' does not have any positive root, then $f' > 0$. Since $f(0) = -2$ and $f(\infty) = \infty$, it follows that f must have a unique zero which in turn is the unique maxima of $H(c)$.

Now having proved that there is a unique maxima, we need to check the second condition from (4.4). That boils down to

$$\psi_c(x) \geq 0$$

on $[-2,2]$ with strict inequality on a dense set. This is equivalent to

$$a_{2n} \geq -a_{2m}c^{2m-2n}\frac{\sum_{l=0}^{m-1}\binom{2l}{l}x^{2(m-l-1)}}{\sum_{l=0}^{n-1}\binom{2l}{l}x^{2(n-l-1)}}$$

for all $x \in [-2,2]$ which in turn is satisfied if and only if

$$a_{2n} \geq -a_{2m}c^{2m-2n}K_{nm}$$

where K_{nm} is defined by (70). On the other hand from the critical Eq. (71) replacing a_{2n}, we arrive at

$$\frac{2 - a_{2m}c^{2m}\binom{2m}{m}}{\binom{2n}{n}} \geq -a_{2m}c^{2m}K_{nm}$$

and thus, after noting that $K_{nm} \leq \binom{2m}{m}/\binom{2n}{n}$, is the same as

$$c \leq \left(\frac{2}{a_{2m}(\binom{2m}{m} - K_{nm}\binom{2n}{n})}\right)^{1/(2m)}.$$

For the function f, we know that $f(x) \leq 0$ if and only if $x \leq c$. Thus, we have the second condition in Corollary 4.4 satisfied if and only if

$$f\left(\left(\frac{2}{a_{2m}(\binom{2m}{m} - K_{nm}\binom{2n}{n})}\right)^{1/(2m)}\right) \geq 0$$

which is equivalent to Eq. (69).

The constant K_{nm} from (70) depends only on n and m. It can be explicitly computed in the case $n = 1$ and any $m \geq 2$ as the minimizer is $t = 0$ and thus $K_{1m} = \binom{2m-2}{m-2}/\binom{2m-2}{m-1}$ and then a simple rearrangement reveals that

$$C_{1m} = \frac{\binom{2m-2}{m-1}}{\binom{2m-2}{m}^{m-1}}.$$

In general, it does not seem that one can find an explicit algebraic expression of the minimizer in (70). For the case of $n = 2$ and $m = 3$, we have an exact solution as the minimizer in the expression there is $t = -2 + \sqrt{6}$ and then in this case $K_{23} = 2\sqrt{6} - 2$ which produces

$$C_{23} = \sqrt{4 + \sqrt{6}}.$$

The root c from Eq. (71) does not have a simple representation in general. However, in some cases it can be solved explicitly. For example if $m = 2n$, one has

$$c = \left(\frac{-a_{2n}\binom{2n}{n} + \sqrt{a_{2n}^2\binom{2n}{n}^2 + 8a_{4n}\binom{4n}{2n}}}{2a_{4n}\binom{4n}{2n}}\right)^{\frac{1}{2n}}$$

and similarly there are algebraic expressions in the case $m = 3n$ or $m = 3k, n = 2k$ and also $m = 4n$ or $m = 4k, n = 3k$, but we omit the lengthy formulae here.

Corollary 5.4. *For the quartic potential*

$$V(x) = a_2\frac{x^2}{2} + a_4\frac{x^4}{4}$$

the equilibrium measure has a single interval support if and only if $a_2 \geq -2\sqrt{a_4}$ *in which case*

$$\begin{aligned} h(x) &= \frac{1}{2\pi}\mathbb{1}_{[-2c,2c]}(x)\sqrt{4c^2 - x^2}(b_2 + a_4x^2) \\ c &= \sqrt{\frac{-a_2 + \sqrt{a_2^2 + 12a_4}}{6a_4}} \\ b_2 &= \frac{2a_2 + \sqrt{a_2^2 + 12a_4}}{3} \end{aligned}$$

$$I_{\mathrm{V}} = \frac{3}{8} + \frac{1}{2}\log\left(\frac{a_2+\sqrt{a_2^2+12a_4}}{2}\right) + \frac{-a_2^4-36a_2^2a_4+162a_4^2+(a_2^3+30a_2a_4)\sqrt{a_2^2+12a_4}}{432a_4^2}$$

We should point out that this example appears for instance in [23].

6. Matching Formal and Analytic Matrix Models

In this section we will prove Theorem 1.2. Our first task is to match the analytic Eq. (59) of (b,c) of a 1-cut potential with the Eq. (13) for $(\mathcal{R},\mathcal{S})$. Consider a 1-cut potential V and its Taylor series expansion at $x=0$:

$$V(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n}.$$

Using the *key identity*

$$\int_{-2}^{2} \frac{x^n \mathrm{d}x}{\pi\sqrt{4-x^2}} = \begin{cases} \binom{n}{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \tag{72}$$

and interchanging summation and integration, (54) gives

$$\int_{-2}^{2} cxV'(cx+b)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = \sum_{n\geq 1} a_n \sum_{j\geq 1}\binom{n-1}{2j-1}\binom{2j}{j}c^{2j}b^{n-2j}$$

$$\int_{-2}^{2} V'(cx+b)\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = \sum_{n\geq 1} a_n \sum_{j\geq 0}\binom{n-1}{2j}\binom{2j}{j}c^{2j}b^{n-2j-1}$$

Then, Eq. (59) gives the system of non-linear equations for (b,c)

$$\begin{cases} \sum_{n\geq 1} a_n \sum_{j\geq 1}\binom{n-1}{2j-1}\binom{2j}{j}c^{2j}b^{n-2j} = 2 \\ \sum_{n\geq 1} a_n \sum_{j\geq 0}\binom{n-1}{2j}\binom{2j}{j}c^{2j}b^{n-2j-1} = 0 \end{cases} \tag{73}$$

Following the notation of [4], let us use the change of variables $(b,c^2) = (S,R)$ as in Eq. (8). Then, (R,S) satisfy the system of equations

$$\begin{cases} \sum_{n\geq 1} a_n \sum_{j\geq 1}\binom{n-1}{2j-1}\binom{2j}{j}R^jS^{n-2j} = 2 \\ \sum_{n\geq 1} a_n \sum_{j\geq 0}\binom{n-1}{2j}\binom{2j}{j}R^jS^{n-2j-1} = 0 \end{cases} \tag{74}$$

Consider now the 1-cut potential

$$V(x) = \frac{x^2}{2} - \sum_{n=1}^{\infty} a_n \frac{x^n}{n}$$

Then, Eq. (74) gives the system of non-linear equations for (R, S)

$$\begin{cases} 2R = 2 + \sum_{n\geq 1} a_n \sum_{j\geq 1} \binom{n-1}{2j-1}\binom{2j}{j} R^j S^{n-2j} \\ S = \sum_{n\geq 1} a_n \sum_{j\geq 0} \binom{n-1}{2j}\binom{2j}{j} R^j S^{n-2j-1} \end{cases} \tag{75}$$

Using

$$\binom{n-1}{2j-1}\binom{2j}{j} = 2\binom{n-1}{j-1}\binom{n-j}{j}$$

it follows that (R, S) satisfy the system of non-linear equations (13).

Observe that for a fixed admissible potential V, Eq. (13) may have none or more than one real solutions for (R, S) but for small parameters $\mathbf{a} = (a_1, a_2, \ldots)$ in some ℓ^1_r for small enough r, R and S become analytic functions of $\mathbf{a}$ (see Theorem 12.1). *However*, it always has a unique formal solution $(\mathcal{R}, \mathcal{S}) \in (1 + \mathcal{A}^+, \mathcal{A}^+)$.

This proves that $\mathcal{R} = R$ and $\mathcal{S} = S$ in Theorem 1.2.

To finish the proof of Theorem 1.2, we need to prove that the coefficient of any monomial $a_1^{n_1} \ldots a_k^{n_k}$ from the power series $\mathcal{F}_0$ and F_0 are equal. The important point here is the fact that the each such monomial involves finitely many $a_1, a_2, \ldots, a_k$ and thus we may assume that all the 1-cut potential is actually a polynomial.

Now, assume that a_n are all 0 for $n \geq k$ and consider potentials of the form

$$V(x) = \frac{x^2}{2} - \left(\sum_{n=1}^{k} a_n \frac{x^n}{n}\right) + \frac{x^{2k+2}}{2k+2}.$$

For small real parameters $a_1, a_2, \ldots, a_k$ the functions

$$g_N(a_1, a_2, \ldots, a_k) = \frac{1}{N^2} \log \frac{\int_{\mathcal{H}_N} \exp(-N\mathrm{Tr}(V(M)))\mathrm{d}M}{\int_{\mathcal{H}_N} \exp(-N\mathrm{Tr}(M^2/2))\mathrm{d}M}$$

are analytic in $a_1, \ldots, a_k$ on a neighborhood of $0 \in \mathbb{R}^k$. Since the limit g_∞ exists, the limit is going to be also an analytic function in these variables. This means that at the level of power series the coefficients must converge to the coefficients of the limit.

On one hand expanding the g_N in power series, the limiting coefficient of $a_1^{n_1} \cdots a_k^{n_k}$ is exactly the corresponding coefficient from the formal model. On the other hand, the limiting function g_∞ is obtained via the potential theory and using the perturbation theory from Sect. 12, it is easy to see that c, b, the solution of the system (75) are actually analytic functions of $a_1, \ldots, a_k$. In particular it means that the planar limit F_0 is equal to g_∞ from (28) and is analytic, thus concluding the proof. □

Remark 6.1. From now on, whenever we have a formal potential $\mathcal{V} = \frac{x^2}{2} - \sum_{n=1}^{\infty} a_n \frac{x^n}{n}$, we will use b, c as the solution to (73), which has a unique solution in $(c, b) \in (1 + \mathcal{A}^+, \mathcal{A}^+)$.

7. The Planar Limit $\mathcal{F}_0(t)$ in Terms of $\mathcal{R}(t)$ and $\mathcal{S}(t)$

This section is devoted to the proofs of Theorems 1.3 and 1.4. After we discuss the proofs we give a main consequences of these formulae, namely the fact that the planar limit enjoys algebricity in some cases which allows complete description of the asymptotics of the coefficients of $\mathcal{F}_0$.

7.1. Proof of Theorem 1.3

In this section we will prove Theorem 1.3 and the first part of Remark 1.3. In this section, it will be convenient to use the 1-cut potentials $\widetilde{\mathcal{V}}_e$ and $\mathcal{V}_e$ given by

$$\tilde{\mathcal{V}}_{\mathrm{e}}(t,x) = \frac{x^2}{2t} - \sum_{n\geq 1} \frac{a_n x^n}{n}$$
$$\mathcal{V}_{\mathrm{e}}(t,x) = \frac{x^2}{2} - \sum_{n\geq 1} \frac{a_n t^{n/2} x^n}{n}.$$

For simplicity of notation in this section we will drop the dependence on e from the writing of $\mathcal{V}_{\mathrm{e}}$ and $\tilde{\mathcal{V}}_{\mathrm{e}}$.

We start by setting $c(t), b(t)$, and $\tilde{c}(t), \tilde{b}(t)$ to be the power series solutions to (73) corresponding to potentials $\mathcal{V}$, respectively $\tilde{\mathcal{V}}$. From the fact that $\mathcal{V}(t,x) = \tilde{\mathcal{V}}(t, \sqrt{t}x)$, we easily get that

$$\tilde{c}(t) = \sqrt{t}c(t) \quad \text{and} \quad \tilde{b}(t) = \sqrt{t}b(t). \tag{76}$$

Then $\tilde{\mathcal{V}}(t,x) = \frac{x^2}{2t} - \mathcal{W}(x)$ and the system satisfied by $\tilde{c}(t), \tilde{b}(t)$ is given by

$$\begin{cases} \int_{-2}^{2} \tilde{c}(t) x \tilde{\mathcal{V}}'(t, \tilde{c}(t)x + \tilde{b}(t)) \frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = 2 \\ \int_{-2}^{2} \tilde{\mathcal{V}}'(t, \tilde{c}(t)x + \tilde{b}(t)) \frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = 0. \end{cases} \tag{77}$$

where the derivative $\tilde{\mathcal{V}}'(t,x)$ is taken with respect to x. Set now,

$$\mathcal{I}(t) = -\log \tilde{c}(t) + \int_{-2}^{2} \frac{\tilde{\mathcal{V}}(t, \tilde{c}(t)x + \tilde{b}(t))\mathrm{d}x}{\pi\sqrt{4-x^2}}$$
$$- \int_{0}^{\tilde{c}(t)} s \left[\left(\int_{-2}^{2} \frac{x\tilde{\mathcal{V}}'(t, sx + \tilde{b}(t))\mathrm{d}x}{2\pi\sqrt{4-x^2}} \right)^2 + \left(\int_{-2}^{2} \frac{\tilde{\mathcal{V}}'(t, sx+\tilde{b}(t))\mathrm{d}x}{\pi\sqrt{4-x^2}} \right)^2 \right] \mathrm{d}s.$$

Taking the derivative with respect to t,

$$\mathcal{I}'(t) = -\frac{\tilde{c}'(t)}{\tilde{c}(t)} + \int_{-2}^{2} \frac{(\tilde{c}'(t)x + \tilde{b}(t))\tilde{\mathcal{V}}'(t, \tilde{c}(t)x + \tilde{b}(t))\mathrm{d}x}{\pi\sqrt{4-x^2}}$$
$$+ \int_{-2}^{2} \frac{\dot{\tilde{\mathcal{V}}}(t, \tilde{c}(t)x + \tilde{b}(t))\mathrm{d}x}{\pi\sqrt{4-x^2}}$$

$$-\tilde{c}'(t)\tilde{c}(t)\left[\left(\int_{-2}^{2}\frac{x\tilde{\mathcal{V}}'(t,\tilde{c}(t)x+\tilde{b}(t))\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)^2\right.$$
$$\left.+\left(\int_{-2}^{2}\frac{\tilde{\mathcal{V}}'(t,\tilde{c}(t)x+\tilde{b}(t))\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)^2\right]$$
$$-2b'(t)\int_0^{\tilde{c}(t)}\int_{-2}^{2}\int_{-2}^{2}\frac{s(xy+4)\tilde{\mathcal{V}}'(t,sx+\tilde{b}(t))\tilde{\mathcal{V}}''(t,sy+\tilde{b}(t))}{4\pi^2\sqrt{(4-x^2)(4-y^2)}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$-2\int_0^{\tilde{c}(t)}\int_{-2}^{2}\int_{-2}^{2}\frac{s(xy+4)\tilde{\mathcal{V}}'(t,sx+\tilde{b}(t))\dot{\tilde{\mathcal{V}}}'(t,sy+\tilde{b}(t))}{4\pi^2\sqrt{(4-x^2)(4-y^2)}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s,$$

where $\dot{\tilde{\mathcal{V}}}(t,x)$ is the derivative with respect to t. Since $\dot{\tilde{\mathcal{V}}}(t,x) = -\frac{x^2}{2t^2}$, the system (77) and Lemma 4.6, we can simplify this to

$$\mathcal{I}'(t) = -\frac{1}{2t^2}\int_{-2}^{2}\frac{(\tilde{c}(t)x+\tilde{b}(t))^2\mathrm{d}x}{\pi\sqrt{4-x^2}}$$
$$+\frac{2}{t^2}\int_0^{\tilde{c}(t)}\int_{-2}^{2}\int_{-2}^{2}\frac{s(xy+4)\tilde{\mathcal{V}}'(t,sx+\tilde{b}(t))(sy+\tilde{b}(t))}{4\pi^2\sqrt{(4-x^2)(4-y^2)}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$= -\frac{2\tilde{c}(t)^2+\tilde{b}(t)^2}{2t^2}+\frac{1}{t^2}\int_0^{\tilde{c}(t)}\int_{-2}^{2}\frac{s(sx+2\tilde{b}(t))\tilde{\mathcal{V}}'(t,sx+\tilde{b}(t))}{\pi\sqrt{4-x^2}}\mathrm{d}x\,\mathrm{d}s.$$

Next, observe that for any continuous function $f:[-2c,2c]\to\mathbb{R}$ with $c>0$, one has

$$\int_0^{c}\int_{-2}^{2}\frac{s(sx+2b)f(sx)}{\pi\sqrt{4-x^2}}\mathrm{d}x\,\mathrm{d}s$$
$$\underset{x=cy/s}{=} c\int_0^{c}\int_{-2s/c}^{2s/c}\frac{s(cy+2b)f(cy)}{\pi\sqrt{4s^2-c^2y^2}}\mathrm{d}y\,\mathrm{d}s$$
$$\underset{\text{Fubini}}{=} c\int_{-2}^{2}\int_{c|y|/2}^{c}\frac{s(cy+2b)f(cy)}{\pi\sqrt{4s^2-c^2y^2}}\mathrm{d}s\,\mathrm{d}y$$

$$= c^2 \int_{-2}^{2} \frac{(cy+2b)f(cy)\sqrt{4-y^2}}{4\pi} \mathrm{d}y$$

$$\underset{y=(z-b)/c}{=} \frac{1}{4\pi} \int_{-2c+b}^{2c+b} (z+b)f(z-b)\sqrt{4c^2-(z-b)^2}\mathrm{d}z.$$

Going back to the previous equation we now have

$$\int_0^{\tilde{c}(t)} \int_{-2}^{2} \frac{s(sx+2\tilde{b}(t))\tilde{\mathcal{V}}'(t,sx+\tilde{b}(t))}{\pi\sqrt{4-x^2}} \, \mathrm{d}x \, \mathrm{d}s$$

$$= \frac{1}{4\pi} \int_{-2\tilde{c}(t)+\tilde{b}(t)}^{2\tilde{c}(t)+\tilde{b}(t)} (z+\tilde{b}(t))\tilde{\mathcal{V}}'(t,z)\sqrt{4\tilde{c}(t)^2-(z-\tilde{b}(t))^2}\mathrm{d}z.$$

Take the derivative with respect to t and observe

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-2\tilde{c}(t)+\tilde{b}(t)}^{2\tilde{c}(t)+\tilde{b}(t)} (z+\tilde{b}(t))\tilde{\mathcal{V}}'(t,z)\sqrt{4\tilde{c}(t)^2-(z-\tilde{b}(t))^2}\mathrm{d}z$$

$$= \int_{-2\tilde{c}(t)+\tilde{b}(t)}^{2\tilde{c}(t)+\tilde{b}(t)} \tilde{b}'(t)\tilde{\mathcal{V}}'(t,z)\sqrt{4\tilde{c}(t)^2-(z-\tilde{b}(t))^2}\mathrm{d}z$$

$$+ \int_{-2\tilde{c}(t)+\tilde{b}(t)}^{2\tilde{c}(t)+\tilde{b}(t)} (z+\tilde{b}(t))\tilde{\mathcal{V}}'(t,z)\frac{4\tilde{c}'(t)-\tilde{b}'(t)(\tilde{b}(t)-z)}{\sqrt{4\tilde{c}(t)^2-(z-\tilde{b}(t))^2}}\mathrm{d}z$$

$$+ \int_{-2\tilde{c}(t)+\tilde{b}(t)}^{2\tilde{c}(t)+\tilde{b}(t)} (z+\tilde{b}(t))\dot{\tilde{\mathcal{V}}}'(t,z)\sqrt{4\tilde{c}(t)^2-(z-\tilde{b}(t))^2}\mathrm{d}z$$

$$= \int_{-2\tilde{c}(t)+\tilde{b}(t)}^{2\tilde{c}(t)+\tilde{b}(t)} \tilde{\mathcal{V}}'(t,z)$$

$$\times \frac{-2\tilde{b}(t)^2\tilde{b}'(t)+4\tilde{c}(t)^2\tilde{b}'(t)+4\tilde{c}'(t)\tilde{c}(t)\tilde{b}(t)+z\left(2\tilde{b}'(t)\tilde{b}(t)+4\tilde{c}'(t)c(t)\right)}{\sqrt{4\tilde{c}(t)^2-(z-\tilde{b}(t))^2}}\mathrm{d}z$$

$$- \int_{-2\tilde{c}(t)+\tilde{b}(t)}^{2\tilde{c}(t)+\tilde{b}(t)} (z+\tilde{b}(t))\frac{z}{t^2}\sqrt{4\tilde{c}(t)^2-(z-\tilde{b}(t))^2}\mathrm{d}z.$$

Changing the variable $z = \tilde{c}(t)x + \tilde{b}(t)$ and using the system (77), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-2\tilde{c}(t)+\tilde{b}(t)}^{2\tilde{c}(t)+\tilde{b}(t)} (z + \tilde{b}(t))\tilde{\mathcal{V}}'(t,z)\sqrt{4\tilde{c}(t)^2 - (z - \tilde{b}(t))^2}\mathrm{d}z$$
$$= 4\pi(\tilde{b}(t)\tilde{b}'(t) + 2\tilde{c}(t)\tilde{c}'(t)) - 2\pi\tilde{c}(t)^2(2\tilde{b}(t)^2 + \tilde{c}(t)^2)/t^2.$$

Therefore we arrive at the equation

$$(t^2\mathcal{I}'(t))' = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\tilde{c}(t)^2 - \frac{\tilde{b}(t)^2}{2}\right)$$
$$+(\tilde{b}(t)\tilde{b}'(t) + 2\tilde{c}(t)\tilde{c}'(t)) - \frac{\tilde{c}(t)^2(2\tilde{b}(t)^2 + \tilde{c}(t)^2)}{2t^2}$$
$$= -\frac{\tilde{c}(t)^2(2\tilde{b}(t)^2 + \tilde{c}(t)^2)}{2t^2}.$$

Since $\mathcal{F}_{0,\mathrm{e}}(t) = \frac{3}{4} - \mathcal{I}(t)$, it implies

$$(t^2\mathcal{F}'_{0,\mathrm{e}}(t))' = \frac{2\mathcal{R}_\mathrm{e}(t)\mathcal{S}_\mathrm{e}^2(t) + \mathcal{R}_\mathrm{e}^2(t)}{2}.$$

This is exactly the statement from (20). To prove also the statement from (19), namely, that

$$\mathcal{F}_{0,\mathrm{e}}(t) = \frac{1}{t}\int_0^t \frac{(t-s)(2\mathcal{R}_\mathrm{e}(s)\mathcal{S}_\mathrm{e}^2(s) + \mathcal{R}_\mathrm{e}^2(s) - 1)}{2s}\mathrm{d}s,$$

denote the right hand side by $\mathcal{G}(t)$ and notice that both sides satisfy the same differential equation, namely

$$(t^2\mathcal{G}'(t))' = \frac{2\mathcal{R}_\mathrm{e}(t)\mathcal{S}_\mathrm{e}^2(t) + \mathcal{R}_\mathrm{e}^2(t)}{2}.$$

In addition, a direct check reveals that

$$\mathcal{F}_{0,\mathrm{e}}(0) = \mathcal{G}(0) = 0, \quad \mathcal{F}'_{0,\mathrm{e}}(0) = \mathcal{G}'(0) = a_1^2/2 + a_2/2$$

which actually follows from the fact that $\mathcal{R}(t) = 1 + a_1 t + O(t^2)$ and $\mathcal{S}(t) = \sqrt{t}a_1 + O(t)$ (see for example the formulae in Appendix A).

7.2. Proof of Theorem 1.4

In this section we will prove Theorem 1.4 and the last part of Remark 1.3. It will be convenient to use the 1-cut potentials $\widetilde{\mathcal{V}}_\mathrm{f}$ and $\mathcal{V}_\mathrm{f}$ given by

$$\tilde{\mathcal{V}}_\mathrm{f}(x) = \frac{x^2}{2} - \sum_{n\geq 3}\frac{a_n x^n}{n}$$
$$\mathcal{V}_\mathrm{F}(x) = \frac{\tilde{\mathcal{V}}_\mathrm{f}(\sqrt{t}x)}{t} = \frac{x^2}{2} - \sum_{k\geq 3}\frac{t^{k/2-1}a_k x^k}{k}.$$

As we did in the previous section, for the sake of simplicity we will drop the dependence on f from the notation $\mathcal{V}_\mathrm{f}$ and $\widetilde{\mathcal{V}}_\mathrm{f}$.

Define $c(t), b(t)$ and $\tilde{c}(t), \tilde{b}(t)$ the power series solutions to (73) corresponding to $\mathcal{V}$ and $\tilde{\mathcal{V}}/t$. Then, one can easily check that

$$\tilde{c}(t) = \sqrt{t}c(t), \quad \tilde{b}(t) = \sqrt{t}b(t). \tag{78}$$

The corresponding system of equations for $\tilde{c}(t)$ and $\tilde{b}(t)$ is

$$\begin{cases} \int_{-2}^{2} \tilde{c}(t)x\tilde{\mathcal{V}}'(\tilde{c}(t)x + \tilde{b}(t))\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = 2t \\ \int_{-2}^{2} \tilde{\mathcal{V}}'(\tilde{c}(t)x + \tilde{b}(t))\frac{\mathrm{d}x}{\pi\sqrt{4-x^2}} = 0. \end{cases} \tag{79}$$

Now set $\mathcal{G}_0(t) = -t^2\mathcal{I}_{0,\tilde{\mathcal{V}}/t}$. Thus

$$\begin{aligned} \mathcal{G}_0(t) &= t^2 \log \tilde{c}(t) - t\int_{-2}^{2} \frac{\tilde{\mathcal{V}}(\tilde{c}(t)x + \tilde{b}(t))\mathrm{d}x}{\pi\sqrt{4-x^2}} \\ &\quad + \int_0^{\tilde{c}(t)} s\left[\left(\int_{-2}^{2} \frac{x\tilde{\mathcal{V}}'(sx + \tilde{b}(t))\mathrm{d}x}{2\pi\sqrt{4-x^2}}\right)^2 \right. \\ &\quad \left. + \left(\int_{-2}^{2} \frac{\tilde{\mathcal{V}}'(sx + \tilde{b}(t))\mathrm{d}x}{\pi\sqrt{4-x^2}}\right)^2\right] \mathrm{d}s. \end{aligned}$$

Differentiating this with respect to t and keeping in mind the system (79), we get

$$\begin{aligned} \mathcal{G}_0'(t) &= 2t\log \tilde{c}(t) - \int_{-2}^{2} \frac{\tilde{\mathcal{V}}(\tilde{c}(t)x + \tilde{b}(t))\mathrm{d}x}{\pi\sqrt{4-x^2}} \\ &\quad +2b'(t)\int_0^{\tilde{c}(t)}\int_{-2}^{2}\int_{-2}^{2} \frac{s(xy+4)\tilde{\mathcal{V}}'(sx+\tilde{b}(t))\tilde{\mathcal{V}}''(sy+\tilde{b}(t))}{4\pi^2\sqrt{(4-x^2)(4-y^2)}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{80}$$

Now, taking $\mathcal{U}(x) = \tilde{\mathcal{V}}'(\tilde{c}(t)x + \tilde{b}(t))$ in Lemma 4.6 and a simple change of variables proves that the last term of (80) becomes 0. Thus, we can continue (80) with

$$\mathcal{G}_0'(t) = 2t\log \tilde{c}(t) - \int_{-2}^{2} \frac{\tilde{\mathcal{V}}(\tilde{c}(t)x + \tilde{b}(t))\mathrm{d}x}{\pi\sqrt{4-x^2}}.$$

Differentiating this with respect to t, and using again the equations from (79), we obtain

$$\mathcal{G}_0''(t) = 2\log \tilde{c}(t).$$

In other words, integrating this twice and keeping in mind that $\mathcal{G}_0(0) = \mathcal{G}_0'(0) = 0$, we get

$$\mathcal{G}_0(t) = 2\int_0^t (t-u)\log\tilde{c}(u)\mathrm{d}u.$$

Now, one has to notice that an easy calculation yields,

$$t^2\mathcal{F}_0(t) = \frac{3t^2}{4} - \frac{t^2}{2}\log t + \mathcal{G}_0(t) = 2\int_0^t (t-u)\log c(u)\mathrm{d}u = \int_0^t \log\mathcal{R}_f(u)\mathrm{d}u,$$

from which Theorem 1.4 follows.

7.3. Proof of Proposition 1.8

Part (a) and (b) of Proposition 1.8 follows from Theorems 1.3 and 1.4.

Part (c) and (d) follow from Proposition 1.7.

8. The Planar Limit for Extreme Edge Potentials

In this section we will compute the planar limit for five extreme formal potentials.

8.1. Exact Formulae

Consider the extreme formal potentials $V_{\mathrm{e}}(x), V_{\mathrm{e}}^{\mathrm{ev}}(x) \in \mathbb{Q}[[t^{1/2}]][[x]]$ given by

$$\mathcal{V}_{\mathrm{e}}^{\mathrm{ev}}(x) = \frac{x^2}{2} + \frac{1}{2}\log(1-tx^2) = \frac{x^2}{2} - \sum_{n=1}^{\infty}\frac{t^n x^{2n}}{2n} \tag{81}$$

$$\mathcal{V}_{\mathrm{e}}(x) = \frac{x^2}{2} + \log(1-\sqrt{t}x) = \frac{x^2}{2} - \sum_{n\geq 1}\frac{t^{n/2}x^n}{n} \tag{82}$$

These potentials correspond to counting planar diagrams with even respectively arbitrary valency of the vertices and a fixed number of edges. Their corresponding invariants $b = \mathcal{S}_{\mathrm{e}} = b(t) = \mathcal{S}_{\mathrm{e}}(t)$ is an element of $\mathbb{Q}[[t^{1/2}]]$, while $c = c(t)$ while $\mathcal{S}_{\mathrm{e}} = \mathcal{S}_{\mathrm{e}}(t)$, $\mathcal{R}_{\mathrm{e}} = \mathcal{R}_{\mathrm{e}}(t)$ and $\mathcal{F}_{0,\mathrm{e}} = \mathcal{F}_{0,\mathrm{e}}(t)$ are elements of $\mathbb{Q}[[t]]$. Our next proposition summarizes the algebraic properties of these elements.

Remark 8.1. For simplicity of writing, in this section we will omit the subscript e in writing $\mathcal{R}_{\mathrm{e}}, \mathcal{S}_{\mathrm{e}}, \mathcal{F}_{0,\mathrm{e}}$.

Proposition 8.2. 1. *For the potential* $\mathcal{V}_{\mathrm{e}}^{\mathrm{ev}}$, *we have*

$$\begin{aligned}
\mathcal{S}(t) &= 0\\
\mathcal{R}(t) &= \frac{1+4t-\sqrt{1-8t}}{8t}\\
\mathcal{F}_0(t) &= \frac{1-24t+72t^2-(1-20t)\sqrt{1-8t}}{128t^2} - \frac{3}{8}\log\frac{1-4t+\sqrt{1-8t}}{2}\\
&= \frac{t}{2} + \frac{3t^2}{4} + 2t^3 + 7t^4 + \frac{144t^5}{5} + 132t^6 + \frac{4576t^7}{7} + 3432t^8 + O(t^{10})
\end{aligned} \tag{83}$$

2. *For the potential* $\mathcal{V}_{\mathrm{e}}$*, we have*

$$\begin{aligned}\mathcal{S}(t) &= \frac{1-\sqrt{1-12t}}{6\sqrt{t}}\\ \mathcal{R}(t) &= \frac{1+12t-\sqrt{1-12t}}{18t}\\ \mathcal{F}_0(t) &= \frac{1-36t+162t^2-(1-30t)\sqrt{1-12t}}{216t^2}-\frac{1}{2}\log\frac{1-6t+\sqrt{1-12t}}{2}\\ &= t+\frac{9t^2}{4}+9t^3+\frac{189t^4}{4}+\frac{1458t^5}{5}+\frac{8019t^6}{4}+\frac{104247t^7}{7}+O(t^9)\end{aligned} \tag{84}$$

Proof. Solving the nonlinear system of Eq. (13) for our formal potentials $\mathcal{V}_{\mathrm{e}}(x)$ and $\mathcal{V}_{\mathrm{e}}^{\mathrm{ev}}(x)$ seems at first an impossible task. Instead, we will use the analytic ideas from Sect. 4 to translate this system into a more tractable one.

In Sect. 6 it was shown that Eq. (13) for $(b, c^2) = (\mathcal{S}, \mathcal{R})$ are exactly Eq. (59) for (b, c) in case of admissible analytic potentials. The proof also works for formal potentials, too, such as our potentials $\mathcal{V}_{\mathrm{e}}(x)$ and $\mathcal{V}^{\mathrm{ev}}(x)$. Proposition 4.5 implies that Eq. (59) are the critical point equations for the function $\mathcal{H}(b, c)$ from Eq. (57). The last function can be computed explicitly for the two formal potentials $\mathcal{V}_{\mathrm{e}}(x)$ and $\mathcal{V}^{\mathrm{ev}}(x)$.

To prove part (1) of the proposition, $\mathcal{V}_{\mathrm{e}}^{\mathrm{ev}}$ is even so $b(t) = 0$. Computing the function $\mathcal{H}$ gives

$$\begin{aligned}\mathcal{H}(c) &= \log c - \frac{c^2}{2} - \frac{1}{4}\int_{-2}^{2}\frac{\log(1-tc^2x^2)\mathrm{d}x}{\pi\sqrt{4-x^2}}\\ &= \log c - \frac{c^2}{2} - \frac{1}{2}\log\frac{1+\sqrt{1-4tc^2}}{2}\end{aligned}$$

and thus

$$\mathcal{H}'(c) = \frac{1}{2c}\left(1-2c^2+\frac{1}{\sqrt{1-4tc^2}}\right).$$

The solution c to $\mathcal{H}'(c) = 0$ such that $c(0) = 1$ satisfies a quartic equation

$$4c^4t + c^2(-1-4t) + t + 1 = 0$$

and it is given by

$$c(t) = \sqrt{\frac{1+4t-\sqrt{1-8t}}{8t}}$$

Given $b(t)$ and $c(t)$ together with (19), we get (83).

For part (2) of the proposition, observe that By Eq. (37) we have:

$$\mathcal{H}(b,c) = \log c - \frac{c^2}{2} - \frac{b^2}{4} - \frac{1}{2}\int_{-2}^{2} \frac{\log(1-\sqrt{t}cx-\sqrt{t}b)\mathrm{d}x}{\pi\sqrt{4-x^2}}$$
$$= \log(c) - \frac{c^2}{2} - \frac{b^2}{4} - \frac{1}{2}\log\frac{1-tb+\sqrt{(1-\sqrt{t}b)^2-4tc^2}}{2}.$$

The critical point (b,c) satisfies the system

$$\begin{cases} 1-c^2+\frac{2tc^2}{(1-\sqrt{t}b+\sqrt{(1-\sqrt{t}b)^2-4tc^2})\sqrt{(1-\sqrt{t}b)^2-4tc^2}} = 0 \\ -b+\frac{\sqrt{t}}{\sqrt{(1-\sqrt{t}b)^2-4tc^2}} = 0. \end{cases} \tag{85}$$

Solve for $\sqrt{(1-\sqrt{t}b)^2-4tc^2} = \sqrt{t}/b$ and plug it in the first equation which becomes

$$1-c^2+\frac{2c^2t}{(\sqrt{t}/b)(1-b\sqrt{t}+\sqrt{t}/b)} = 0.$$

In turn, this implies

$$c^2 = \frac{b+\sqrt{t}-b^2\sqrt{t}}{b+\sqrt{t}-3b^2\sqrt{t}}.$$

We need to pick the solution c which for $t=0$ is 1 and thus

$$c = \frac{\sqrt{b+\sqrt{t}-b^2\sqrt{t}}}{\sqrt{b+\sqrt{t}-3b^2\sqrt{t}}}.$$

We go back to the second equation of (85) and solve for c as a function of b to get

$$c^2 = \frac{(1-b\sqrt{t})^2b^2-t}{4tb^2}.$$

Equating now the two expressions of c^2 in terms of b shows that

$$\frac{b+\sqrt{t}-b^2\sqrt{t}}{b+\sqrt{t}-3b^2\sqrt{t}} = \frac{(1-b\sqrt{t})^2b^2-t}{4tb^2}.$$

This implies that b satisfies

$$(-b-\sqrt{t}+b^2\sqrt{t})^2(-b+\sqrt{t}+3b^2\sqrt{t}) = 0.$$

There are four solutions to this equation,

$$\frac{1-\sqrt{1-12t}}{6\sqrt{t}}, \quad \frac{1+\sqrt{1-12t}}{6\sqrt{t}}, \quad \frac{1-\sqrt{1+4t}}{2\sqrt{t}}, \quad \frac{1+\sqrt{1+4t}}{2\sqrt{t}}.$$

Since $b(0)=0$, this eliminates the second and the fourth solutions. To decide which one is the right one, we notice that $c(0)=1$ and this implies

$$b = \frac{1 - \sqrt{1 - 12t}}{6\sqrt{t}} \quad \text{and} \quad c = \sqrt{\frac{1 + 12t - \sqrt{1 - 12t}}{18t}}.$$

Now using (19) one concludes (84). □

8.2. A review of Holonomic Functions and their Asymptotics

In this section we briefly review some standard facts about holonomic functions and their asymptotics from [35]. Recall that a formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{86}$$

is *holonomic* if it satisfies a linear differential equation

$$\sum_{j=0}^{d} c_j(x) f^{(j)}(x) = 0$$

where $c_j(x) \in \mathbb{Q}[x]$ for $j = 0, \ldots, d$ with $c_d(x) \neq 0$. A sequence (a_n) is *holonomic* if it satisfies a linear recursion

$$\sum_{j=0}^{r} \gamma_j(n) a_{n+j} = 0$$

for all $n \in \mathbb{N}$ where $\gamma_j(n) \in \mathbb{Q}[n]$ with $\gamma_r(n) \neq 0$. It is easy to see that a sequence (a_n) is holonomic if and only if the generating series (86) is holonomic. Of importance to us are *algebraic functions* $y = f(x)$ i.e., solutions to polynomial equations

$$\sum_{j=0}^{d} c_j(x) y^j = 0 \tag{87}$$

where $c_j(x) \in \mathbb{Q}[x]$ for $j = 0, \ldots, d$ and $c_d(x) \neq 0$. Algebraic functions regular at $x = 0$ are always holonomic; see for example [9]. Moreover, algebraic functions regular at $x = 0$ have holomorphic extensions to a finite branched cover of the complex plane branched along a finite set of algebraic points, given by the roots of the discriminant of the polynomial (87) with respect to y. Locally, at a point $x = x_0 \in \overline{\mathbb{Q}}$ of the field of algebraic numbers, an algebraic function $y(x)$ has a convergent power series expansion of the form

$$y(x) = \sum_{n=0}^{\infty} c_{n/d} (x - x_0)^{n/d}$$

for some natural number d and for algebraic numbers $c_{n/d}$. This is the content of *Puiseux's theorem* [8,40]. If $y(x)$ is regular at $x = 0$ with Taylor series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

the asymptotics of the sequence (a_n) can be computed explicitly by the singularities of $y(x)$ which are nearest to $x = 0$. The computation also includes the Stokes constants. A computer implementation of the rigorous computation is available from [24]. In fact, (a_n) is a sequence of *Nilsson type* discussed in detail in [20].

8.3. Holonomicity and Asymptotics

In this section we illustrate Proposition 1.8 with the concrete examples of the extreme potentials and study the coefficients of the Taylor series (f_n) of the planar limit written as

$$\mathcal{F}_0(t) = \sum_{n=1}^{\infty} f_n t^n.$$

Proposition 8.3. (1) *For the potential* $\mathcal{V}_{\mathrm{e}}^{\mathrm{ev}}, \mathcal{R}, \mathcal{G}_0 = \mathcal{G}_0(t) = \mathcal{F}_0'(t)$ *and* $\mathcal{F}_0$ *satisfy*

$$\begin{aligned} 4\mathcal{R}^2 t - \mathcal{R}(1+4t) + t + 1 &= 0 \\ 64t^3\mathcal{G}_0^2 + (48t^2 - 24t + 2)\mathcal{G}_0 + 9t - 1 &= 0 \\ 3 + 6(4t-1)\mathcal{F}_0' + 2t(8t-1)\mathcal{F}_0'' &= 0, \quad \mathcal{F}_0(0) = 0,\ \mathcal{F}_0'(0) = 1/2 \\ (n+3)(n+1)f_{n+1} - 4n(1+2n)f_n &= 0, \quad n \geq 1,\ f_1 = 1/2. \end{aligned} \tag{88}$$

In addition,

$$\mathcal{F}_0(t) = \sum_{n\geq 1} \frac{3(2n-1)!2^{n-1}}{n!(n+2)!} t^n, \tag{89}$$

and for large n, *the asymptotics of* f_n *is*

$$f_n = \frac{3}{4\sqrt{\pi}} \frac{8^n}{n^{7/2}} \left(1 - \frac{25}{8n} + \frac{945}{128n^2} - \frac{16275}{1024n^3} + O\left(\frac{1}{n^4}\right)\right). \tag{90}$$

(2) *For the potential* $\mathcal{V}_{\mathrm{e}}$, *we have that* $\mathcal{S}, \mathcal{R}, \mathcal{G}_0 = \mathcal{F}_0'$ *and* $\mathcal{F}_0$ *satisfy*

$$\begin{aligned} 3\sqrt{t}\mathcal{S}^2 - \mathcal{S} + \sqrt{t} &= 0 \\ 9t\mathcal{R}^2 - (12t+1)\mathcal{R} + 1 &= 0 \\ 108t^3\mathcal{G}_0^2 + (108t^2 - 36t + 2)\mathcal{G}_0 + 27t - 2 &= 0 \\ 3 + 3(6t-1)\mathcal{F}_0' + t(12t-1)\mathcal{F}_0'' &= 0, \quad \mathcal{F}_0(0) = 1,\ \mathcal{F}_0'(0) = 1 \\ (n+3)(n+1)f_{n+1} - 6n(1+2n)f_n &= 0, \quad f_1 = 1. \end{aligned} \tag{91}$$

In addition,

$$\mathcal{F}_0(t) = \sum_{n\geq 1} \frac{2(2n-1)!3^n}{n!(n+2)!} t^n, \tag{92}$$

and for large n,

$$f_n = \frac{2}{\sqrt{\pi}} \frac{12^n}{n^{7/2}} \left(1 - \frac{25}{16n} + \frac{945}{256n^2} - \frac{16275}{2048n^3} + O\left(\frac{1}{n^4}\right)\right). \tag{93}$$

Proof. It is straightforward to see that (88) follows from (83) while (91) from (84).

For (1), a direct check proves that $\mathcal{F}_0(t)$ solves the third equation of (88). This immediately implies the recurrence on f_n and then the closed formula in (89) which in turn combined with Stirling's formula leads to (92).

Another way of checking the closed formula (89) is the following. Observe by a direct calculation that

$$(8t^2-t)\mathcal{F}_0'''(t)+(28t-4)\mathcal{F}_0''(t)+12\mathcal{F}_0'(t)=0,$$
$$\mathcal{F}_0(0)=0,\quad \mathcal{F}_0'(0)=1/2,\quad \mathcal{F}_0''(0)=3/2.$$

This is a hypergeometric equation, and its solution is given by

$$\mathcal{F}_0(t)=\frac{1}{2}t\,{}_3F_2(1,1,3/2;2,4;8t)=\sum_{n\ge 1}\frac{3(2n-1)!2^{n-1}}{n!(n+2)!}t^n,$$

where ${}_3F_2(a_1,a_2,a_3;b_1,b_2;x)$ stands for the hypergeometric function with parameters a_1,a_2,a_3 and b_1,b_2. This is exactly (89). Using *Stirling's formula* (see [32]), one can easily deduce (89).

For (2), use the same proof as for (1). □

8.4. Three More Flavors of The Extreme Edge Potentials

In this section we will investigate the following three flavors of the extreme edge potentials (82) and (81), given by

$$\mathcal{V}_1(x)=\frac{(1+t)x^2}{2}+\frac{1}{2}\log(1-tx^2)=\frac{x^2}{2}-\sum_{n\ge 2}\frac{t^n x^{2n}}{2n} \tag{94}$$

$$\mathcal{V}_2(x)=\frac{x^2}{2}+\sqrt{t}x+\log(1-\sqrt{t}x)=\frac{x^2}{2}-\sum_{n\ge 2}\frac{t^{n/2}x^n}{n} \tag{95}$$

$$\mathcal{V}_3(x)=\frac{(1+t)x^2}{2}+\sqrt{t}x+\log(1-\sqrt{t}x)=\frac{x^2}{2}-\sum_{n\ge 3}\frac{t^{n/2}x^n}{n}. \tag{96}$$

These correspond to the counting of planar diagrams with a fixed number of edges and vertices of even valency greater or equal to 4, or arbitrary valency greater or equal to 2, respectively arbitrary valency greater or equal to 3.

Proposition 8.4. (1) *For the potential $\mathcal{V}_1$, we have*

$$\mathcal{S}(t)=0 \tag{97}$$

$$\mathcal{R}(t)=\frac{1+5t-\sqrt{(1+t)(1-7t)}}{8t(1+t)} \tag{98}$$

$$\mathcal{F}_0(t)=\frac{1-22t+49t^2-(1-19t)\sqrt{(1+t)(1-7t)}}{128t^2}-\frac{1}{8}\log(1+t) \tag{99}$$
$$-\frac{3}{8}\log\frac{1-3t+\sqrt{(1+t)(1-7t)}}{2}$$
$$=\frac{t^2}{2}+\frac{5t^3}{6}+\frac{23t^4}{8}+\frac{51t^5}{5}+\frac{124t^6}{3}+\frac{2515t^7}{14}+\frac{13245t^8}{16}+O(t^9). \tag{100}$$

(2) *For the potential* $\mathcal{V}_2$*, we have*

$$\mathcal{S}(t) = \frac{1 - 5t - \sqrt{1 - 10t + t^2}}{6\sqrt{t}} \tag{101}$$

$$\mathcal{R}(t) = \frac{1 + 14t + t^2 - (1+t)\sqrt{1 - 10t + t^2}}{18t} \tag{102}$$

$$\mathcal{F}_0(t) = \frac{1 - 32t + 96t^2 + 76t^3 + t^4 - (1 - 27t - 27t^2 + t^3)\sqrt{1 - 10t + t^2}}{216t^2} - \log\frac{1 + t + \sqrt{1 - 10t + t^2}}{2} \tag{103}$$

$$= \frac{t}{2} + \frac{3t^2}{4} + \frac{8t^3}{3} + 12t^4 + \frac{312t^5}{5} + \frac{1076t^6}{3} + \frac{15528t^7}{7} + 14508t^8 + O(t^9). \tag{104}$$

(3) *For the potential* $\mathcal{V}_3$*, we have*

$$\mathcal{S}(t) = \frac{1 - 4t - \sqrt{1 - 8t - 8t^2}}{6(1+t)\sqrt{t}} \tag{105}$$

$$\mathcal{R}(t) = \frac{1 + 16t + 16t^2 - (1+16t)\sqrt{1 - 8t - 8t^2}}{18t(1+t)^2} \tag{106}$$

$$\mathcal{F}_0(t) = \frac{1 - 28t + 6t^2 + 176t^3 + 142t^4 - (1 - 24t - 78t^2 - 52t^3)\sqrt{1 - 8t - 8t^2}}{216t^2(1+t)^2} + \frac{1}{2}\log(1+t) - \log\frac{1 + 2t + \sqrt{1 - 8t - 8t^2}}{2} \tag{107}$$

$$= \frac{t^2}{2} + \frac{3t^3}{2} + \frac{47t^4}{8} + \frac{139t^5}{5} + \frac{430t^6}{3} + \frac{11175t^7}{14} + \frac{75149t^8}{16} + O(t^9). \tag{108}$$

Proof. We follow the same approach as in Proposition 8.2.

For (1), the function $\mathcal{H}$ becomes

$$\mathcal{H}(c) = \log c - \frac{(1+t)c^2}{2} - \frac{1}{4}\int_{-2}^{2} \frac{\log(1 - c^2tx^2)\mathrm{d}x}{\pi\sqrt{4 - x^2}}$$
$$= \log c - \frac{(1+t)c^2}{2} - \frac{1}{2}\log\frac{1 + \sqrt{1 - 4c^2t}}{2}$$

and thus

$$\mathcal{H}'(c) = \frac{1}{2c}\left(1 - 2(1+t)c^2 + \frac{1}{\sqrt{1 - 4c^2t}}\right).$$

The solution to $\mathcal{H}'(c) = 0$ with $c(0) = 1$ is

$$c(t) = \frac{\sqrt{1 + 5t - \sqrt{(1+t)(1-7t)}}}{2\sqrt{2t(1+t)}}.$$

From this, using (19), gives (99).

For (2), we have

$$\mathcal{H}(b,c) = \log c - \frac{c^2}{2} - \frac{b^2}{4} - \frac{b\sqrt{t}}{2} - \frac{1}{2}\int_{-2}^{2} \frac{\log(1-\sqrt{t}cx-\sqrt{t}b)\mathrm{d}x}{\pi\sqrt{4-x^2}}$$

$$= \log(c) - \frac{c^2}{2} - \frac{b^2}{4} - \frac{b\sqrt{t}}{2} - \frac{1}{2}\log\frac{1-\sqrt{t}b+\sqrt{(1-\sqrt{t}b)^2-4tc^2}}{2}.$$

The critical point (b,c) satisfies the system

$$\begin{cases} 1-c^2+\frac{2tc^2}{(1-\sqrt{t}b+\sqrt{(1-\sqrt{t}b)^2-4tc^2})\sqrt{(1-\sqrt{t}b)^2-4tc^2}} = 0 \\ -b-\sqrt{t}+\frac{\sqrt{t}}{\sqrt{(1-\sqrt{t}b)^2-4tc^2}} = 0. \end{cases}$$

The solution to this system such that $c(0)=1$ is given by

$$c(t) = \frac{\sqrt{1+14t+t^2-(1+t)\sqrt{1-10t+t^2}}}{3\sqrt{2t}}$$

and

$$b(t) = \frac{1-5t-\sqrt{1-10t+t^2}}{6\sqrt{t}}$$

Then (19) together with some simplifications give (103).

For (3) we have

$$\mathcal{H}(b,c) = \log c - \frac{(1+t)c^2}{2} - \frac{(1+t)b^2}{4} - \frac{b\sqrt{t}}{2} - \frac{1}{2}\int_{-2}^{2} \frac{\log(1-\sqrt{t}cx-\sqrt{t}b)\mathrm{d}x}{\pi\sqrt{4-x^2}}$$

$$= \log(c) - \frac{(1+t)c^2}{2} - \frac{(1+t)b^2}{4} - \frac{b\sqrt{t}}{2}$$
$$-\frac{1}{2}\log\frac{1-\sqrt{t}b+\sqrt{(1-\sqrt{t}b)^2-4tc^2}}{2}.$$

The critical point (b,c) satisfies the system

$$\begin{cases} 1-c^2+\frac{2tc^2}{(1-\sqrt{t}b+\sqrt{(1-\sqrt{t}b)^2-4tc^2})\sqrt{(1-\sqrt{t}b)^2-4tc^2}} = 0 \\ -(1+t)b-\sqrt{t}+\frac{\sqrt{t}}{\sqrt{(1-\sqrt{t}b)^2-4tc^2}} = 0, \end{cases}$$

with the solution satisfying $c(0)=1$, being

$$c(t) = \frac{\sqrt{1+16t+16t^2-(1+16t)\sqrt{1-8t-8t^2}}}{3(1+t)\sqrt{2t}}$$

and

$$b(t) = \frac{1-4t-\sqrt{1-8t-8t^2}}{6(1+t)\sqrt{t}}.$$

Finally, using (19) one obtains (107). □

Next we present algebraic and differential equations satisfied by $\mathcal{G}_0 = \mathcal{F}_0'(t)$ and the recursion relation for the Taylor coefficients (f_n) of $\mathcal{F}_0(t)$ and their exact asymptotic expansions.

Proposition 8.5. (1) *For the potential $\mathcal{V}_1, \mathcal{G}_0$ satisfies*

$$-2t + 13t^2 + 16t^3 + 2(1 - 9t + 3t^2 + 45t^3 + 32t^4)\mathcal{G}_0 + 64t^3(1+t)^2\mathcal{G}_0^2 = 0,$$

and $\mathcal{F}_0(t)$ satisfies

$$t(8+7t) - 2(3 - 4t - 21t^2 - 14t^3)\mathcal{F}_0' - 2t(1 - 5t - 13t^2 - 7t^3)\mathcal{F}_0'' = 0, \tag{109}$$

with $\mathcal{F}_0(0) = 0, \mathcal{F}_0'(0) = 0$ and (f_n) satisfies

$$\begin{aligned}49n^2(1+n)f_n + 7(1+n)^2(32+21n)f_{n+1} + (2+n)(544+543n+139n^2)f_{n+2} \\ +(3+n)(224+157n+33n^2)f_{n+3} - 8(2+n)(4+n)(6+n)f_{n+4} = 0,\end{aligned}$$

with $f_1 = 0, f_2 = 1/2, f_3 = 5/6$. For large n, the asymptotics of (f_n) is given by

$$f_n = \frac{147}{512}\sqrt{\frac{7}{2\pi}}\frac{7^n}{n^{7/2}}\left(1 - \frac{105}{32n} + \frac{16065}{2048n^2} - \frac{1109115}{65536n^3} + O\left(\frac{1}{n^4}\right)\right). \tag{110}$$

(2) *For the potential $\mathcal{V}_2, \mathcal{G}_0$ satisfies*

$$\begin{aligned}1 - 13t + 22t^2 - 9t^3 - t^4 + (-2 + 32t - 108t^2 + 76t^3 + 2t^4)\mathcal{G}_0 \\ -108t^3\mathcal{G}_0^2 = 0\end{aligned} \tag{111}$$

and $\mathcal{F}_0(t)$ satisfies

$$3 - 5t + t^2 + t^3 - 2(3 - 17t + 5t^2 + t^3)\mathcal{F}_0' - 2t(1 - 11t + 11t^2 - t^3)\mathcal{F}_0'' = 0,$$

with $\mathcal{F}_0(0) = 0, \mathcal{F}_0'(0) = 1/2$ and (f_n) satisfies

$$\begin{aligned}&(-2+n)(-1+n)nf_n - 2(-1+n)(1+n)(2+5n)f_{n+1} \\ &-(2+n)(44+25n+5n^2)f_{n+2} + 4(3+n)(116+91n+17n^2)f_{n+3} \\ &-(4+n)(1326+701n+89n^2)f_{n+4} + 2(5+n)(6+n)(85+19n)f_{n+5} \\ &-3(5+n)(6+n)(8+n)f_{n+6} = 0,\end{aligned}$$

with $f_1 = 1/2, f_2 = 3/4, f_3 = 8/3, f_4 = 12, f_5 = 312/5$. Moreover, for large n, the asymptotics of (f_n) is given by

$$f_n = \frac{2}{3\sqrt{\pi}}\sqrt[4]{\frac{2}{3}}\frac{(5+2\sqrt{6})^n}{n^{7/2}}\left(1 - \frac{45\sqrt{6}}{32n} + \frac{8435}{1024n^2} - \frac{238805\sqrt{6}}{32768n^3} + O\left(\frac{1}{n^4}\right)\right). \tag{112}$$

(3) *For the potential $\mathcal{V}_3, \mathcal{G}_0$ satisfies*

$$\begin{aligned}&-2t + 11t^2 + 65t^3 + 107t^4 + 81t^5 + 27t^6 + (2 - 20t - 22t^2 + 184t^3 + 560t^4 \\ &+700t^5 + 432t^6 + 108t^7)\mathcal{G}_0 + (108t^3 + 540t^4 + 1080t^5 + 1080t^6 + 540t^7 \\ &+108t^8)\mathcal{G}_0^2 = 0\end{aligned} \tag{113}$$

and $\mathcal{F}_0(t)$ *satisfies*

$$-8t-35t^2-44t^3-16t^4-6(-1+15t^2+34t^3+28t^4+8t^5)\mathcal{F}_0' \\ -2(-t+5t^2+29t^3+47t^4+32t^5+8t^6)\mathcal{F}_0''=0,$$

with $\mathcal{F}_0(0)=0, \mathcal{F}_0'(0)=0$ *and* (f_n) *satisfies*

$$\begin{aligned}&128n^2(2+n)f_n+32(1+n)(66+97n+27n^2)f_{n+1}\\ &\quad+8(2+n)(1836+1568n+305n^2)f_{n+2}\\ &\quad+4(3+n)(9972+6193n+929n^2)f_{n+3}\\ &\quad+(4+n)(54276+26661n+3257n^2)f_{n+4}\\ &\quad+(5+n)(38220+15479n+1595n^2)f_{n+5}\\ &\quad+3(6+n)(3972+1349n+121n^2)f_{n+6}\\ &\quad+5(7+n)(36+5n+n^2)f_{n+7}-8(6+n)(8+n)(10+n)f_{n+8}=0\end{aligned}$$

with $f_1=0, f_2=1/2, f_3=3/2, f_4=47/8, f_5=138/5, f_6=430/3, f_7=11175/14$. *Moreover, for large* n*, the asymptotics of* (f_n) *is given by*

$$f_n=c\frac{(4+2\sqrt{6})^n}{n^{7/2}}\left(1-\frac{5\left(62-23\sqrt{6}\right)}{8n}+\frac{35\left(4567-1858\sqrt{6}\right)}{64n^2}\right. \\ \left.-\frac{35\left(2608410-1064767\sqrt{6}\right)}{512n^3}+O\left(\frac{1}{n^4}\right)\right), \tag{114}$$

with

$$c=\frac{32}{3\sqrt{(267+109\sqrt{6})\pi}}.$$

Remark 8.6. Notice here that the exponential rates change depending on the counting problem at hand. Excluding just a few types of vertices leads to different exponential behavior. It is worth pointing out that in

$$f_n\sim C\frac{{t_0}^{-n}}{n^{3-\gamma}}$$

although the exponential growth rate t_0 depends on the details of the model, the exponent γ is universal, as was observed in [27] and also in [3, Sect. 2].

The recurrence relations for the coefficients (f_n) are not in general of the lowest degree. However we did not attempt to simplify them even further because they are easily deduced from the differential equations satisfied by $\mathcal{F}_0$.

Remark 8.7. The linear recursions for (f_n) or the linear differential equation for $\mathcal{F}_0$ cannot compute the Stokes constants, i.e., the leading terms in the asymptotic expansion (110).

It is the algebricity of $\mathcal{F}_0'(t)$ which uniquely determines the Stokes constants. In the case at hand the Stokes constants come from the explicit expressions of $\mathcal{F}_0$.

Proof. The results from Eqs. (109), (111) and (113) follow straightforwardly from Eqs. (99), (103) and (107).

The idea of proving these asymptotics is based on the analysis of singularities as explained in [19]. This is particularly tractable as we have explicit expressions for the planar limits.

For $\mathcal{V}_1$, from (99) one can see that the singularities of $\mathcal{F}_0$ are at $t_0 = 1/7$ and $t_0 = -1$. The smallest singularity gives the leading terms. In our case the expansion of $\mathcal{F}_0$ near $t_0 = 1/7$ is

$$\frac{1}{16}(-7 - 12\log(2) + 8\log(7)) - \frac{3(1-7t)}{64} + \frac{137(1-7t)^2}{1024} - \frac{49}{320}\sqrt{\frac{7}{2}}(1-7t)^{5/2} + \frac{5569(1-7t)^3}{12288} - \frac{343\sqrt{\frac{7}{2}}(1-7t)^{7/2}}{1024} + \frac{105473(1-7t)^4}{131072} - \frac{51793\sqrt{\frac{7}{2}}(1-7t)^{9/2}}{98304} + O((1-7t)^5).$$

Since the main contribution to the asymptotics of the coefficients is given by the half powers, combined with

$$(1-x)^\alpha = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}x^k \tag{115}$$

applied for $\alpha = 5/2, 7/2, 9/2$, after a simple asymptotics expansion give the result of (110).

For $\mathcal{V}_2$, the same argument works in this case for the other examples. Using (103), we can see that the singularities of $\mathcal{F}_0$ are $t_0 = 5 - 2\sqrt{6}$ and $5 + 2\sqrt{6}$. For the asymptotics of the coefficients the leading one is the smallest, namely $t_0 = 5 - 2\sqrt{6}$. The expansion near t_0 is in this case

$$\left(\frac{23}{12} - \sqrt{6} - \log(3-\sqrt{6})\right) + \left(-2 + \frac{7\sqrt{\frac{2}{3}}}{3}\right)\left(1 - \frac{t}{t_0}\right) + \frac{1}{36}(15 - 4\sqrt{6})\left(1-\frac{t}{t_0}\right)^2 - \frac{16}{45}\left(\frac{2}{3}\right)^{1/4}\left(1-\frac{t}{t_0}\right)^{5/2} + \frac{1}{27}\left(9 + 2\sqrt{6}\right)\left(1-\frac{t}{t_0}\right)^3 - \frac{1}{63}\sqrt{1008 + 1270\sqrt{\frac{2}{3}}}\left(1-\frac{t}{t_0}\right)^{7/2} + \left(\frac{14}{27} + \frac{1}{\sqrt{6}}\right)\left(1-\frac{t}{t_0}\right)^4 - \frac{\sqrt{1089936 + 1337137\sqrt{\frac{2}{3}}}\left(1-\frac{t}{t_0}\right)^{9/2}}{1296} + O\left(\left(1-\frac{t}{t_0}\right)^5\right).$$

Considering the coefficients of the half powers, and noting that $1/t_0 = (5 + 2\sqrt{6})$, one gets (112).

For $\mathcal{V}_3$, we proceed similarly. From (107), observe that the singularities of $\mathcal{F}_0$ are $t_0 = \frac{1}{4}(-2+\sqrt{6})$ and $\frac{1}{4}(-2-\sqrt{6})$. The one with smallest absolute value is $t_0 = \frac{1}{4}(-2+\sqrt{6})$. The expansion near this point is given by

$$\left(\frac{23}{12} - \sqrt{6} + \frac{1}{2}\log\left(\frac{2}{3}\left(2+\sqrt{6}\right)\right)\right) + \frac{1}{18}\left(357 - 146\sqrt{6}\right)\left(1 - \frac{t}{t_0}\right)$$
$$+ \left(\frac{2539}{12} - 259\sqrt{\frac{2}{3}}\right)\left(1 - \frac{t}{t_0}\right)^2 - \frac{256\left(1-\frac{t}{t_0}\right)^{5/2}}{45\sqrt{267+109\sqrt{6}}}$$
$$+ \frac{1}{54}\left(122589 - 50038\sqrt{6}\right)\left(1 - \frac{t}{t_0}\right)^3 - \frac{1472\left(1-\frac{t}{t_0}\right)^{7/2}}{21\sqrt{7929+3237\sqrt{6}}}$$
$$+ \frac{1}{216}\left(5233579 - 2136536\sqrt{6}\right)\left(1 - \frac{t}{t_0}\right)^4 - \frac{16520\left(1-\frac{t}{t_0}\right)^{9/2}}{81\sqrt{26163+10681\sqrt{6}}}$$
$$+ O\left(\left(1 - \frac{t}{t_0}\right)^5\right)$$

from which, noticing that $1/t_0 = 4 + 2\sqrt{6}$ one can deduce (114). □

9. The Planar Limit for Extreme Face Potentials

Consider the extreme formal potentials

$$\mathcal{V}_{\mathrm{f}}^{\mathrm{ev}}(x) = \frac{x^2}{2} - \sum_{n\geq 4} \frac{t^{n-1}x^{2n}}{2n} \tag{116}$$

$$\mathcal{V}_{\mathrm{f}}(x) = \frac{x^2}{2} - \sum_{n\geq 3} \frac{t^{n/2-1}x^{n}}{n}. \tag{117}$$

We consider now the case of extremal potentials and compute the corresponding planar limit.

Remark 9.1. For simplicity, in this section we will drop the subscript f from the writings of $\mathcal{R}_{\mathrm{f}}, \mathcal{F}_{0,\mathrm{f}}, \mathcal{V}_{\mathrm{f}}$.

Proposition 9.2. (1) *For the potential $\mathcal{V}^{\mathrm{ev}}$, the planar limit $\mathcal{F}_0(t)$ has the following ten terms in the Taylor expansion*

$$\mathcal{F}_0(t) = \frac{t}{2} + \frac{47t^2}{24} + \frac{49t^3}{4} + \frac{11839t^4}{120} + \frac{9283t^5}{10} + \frac{3260543t^6}{336}$$
$$+ \frac{183387797t^7}{168} + \frac{941448191t^8}{720}$$
$$+ \frac{490223647t^9}{30} + \frac{93171535189t^{10}}{440} + O(t^{11})$$

and its radius of convergence is $t_0 = \frac{4-3\sqrt[3]{2}}{4}$. In addition, if f_n is the coefficient of t^n in the Taylor expansion of F_0, then the asymptotic is

$$f_n = \frac{1}{3}\sqrt{\frac{2\sqrt[3]{2}-1}{\pi}}\frac{\left(\frac{4}{4-3\sqrt[3]{2}}\right)^n}{n^{7/2}} \times \left(1 - \frac{243-8\sqrt[3]{2}}{72n} + \frac{91881-2640\sqrt[3]{2}-5696\sqrt[3]{4}}{10368n^2} + O\left(\frac{1}{n^3}\right)\right). \tag{118}$$

(2) *For the potential* $\mathcal{V}$*, the planar limit* $\mathcal{F}_0$ *has the Taylor expansion*

$$\mathcal{F}_0(t) = \frac{7t}{6} + \frac{109t^2}{8} + \frac{15631t^3}{60} + \frac{256629t^4}{40} + \frac{38720767t^5}{210} + \frac{658811733t^6}{112} + O(t^7).$$

The planar limit has radius of convergence given by t_0*, the only positive root of the polynomial equation*

$$-11 - 128t + 41088t^2 - 20480t^3 + 4096t^4 = 0.$$

The coefficient f_n *of* t^n *from the expansion of* $\mathcal{F}_0$ *has the asymptotic expansion*

$$f_n \sim c\frac{(1/t_0)^n}{n^{7/2}}\left(1 + \frac{d_1}{n} + \frac{d_2}{n^2} + O\left(\frac{1}{n^3}\right)\right), \tag{119}$$

with

$$c = \sqrt{\frac{34133 - 914556t_0 + 449856t_0^2 - 89344t_0^3}{176868\pi}}$$

$$d_1 = -\frac{36145645 + 79913928t_0 - 39094848t_0^2 + 7808512t_0^3}{11319552}$$

$$d_2 = \frac{7806311269 + 20984001752t_0 - 10129539392t_0^2 + 2006727168t_0^3}{1026306048}$$

Numerically,

$$t_0 = 0.0180827901833\dots$$
$$1/t_0 = 55.3012001942\dots$$
$$c = 0.1786898225\dots$$
$$d_1 = -3.3197404318\dots$$
$$d_2 = 7.9727292073\dots.$$

Proof. For part (1), we will find the singularities of $\mathcal{R}(t) = c^2(t)$ and then expand it around its singularities nearest to the origin, as explained in [20].

First, notice that

$$\mathcal{H}(c) = \log(c) - c^2 - \frac{1}{2t}\log\frac{1+\sqrt{1-4tc^2}}{2}$$

and $\mathcal{H}'(c) = 0$ implies that the equation satisfied by $\mathcal{R} = c^2$ is

$$1 - \mathcal{R} - t^2 + 4\mathcal{R}^2t^2 + 4\mathcal{R}t^4 - 16\mathcal{R}^2t^4 + 16\mathcal{R}^3t^4 = 0. \tag{120}$$

Our condition is that $\mathcal{R}(0) = 1$ and this determines the branch near 0. One can actually find the solution explicitly, however that is not very useful. The singularities of $\mathcal{R}$ are at the points where the discriminant of (120) vanishes. That means that the singularities are solutions to the polynomial equation

$$-16(-5t^2 + 96t^3 - 96t^4 + 32t^5) = 0.$$

The latter are given by

$$\left\{0, \frac{1}{2}\left(2 - \frac{3}{2^{2/3}}\right), 1 + \frac{3\left(1 - i\sqrt{3}\right)}{42^{2/3}}, 1 + \frac{3\left(1 + i\sqrt{3}\right)}{42^{2/3}}\right\}.$$

The singularity of $\mathcal{R}$ is thus $t_0 = \frac{1}{2}\left(2 - \frac{3}{2^{2/3}}\right) \approx 0.0550592$. The series expansion of $\mathcal{R}$ near t_0 is

$$\begin{aligned}
\mathcal{R}(t) = \frac{1}{10}\left(7 + 4\sqrt[3]{2} + 3\sqrt[3]{4}\right) - \frac{104976}{1574640}\sqrt{15\left(9 + 8\sqrt[3]{2} + 6\sqrt[3]{4}\right)}\left(1 - \frac{t}{t_0}\right)^{1/2} \\
+\frac{17496}{1574640}\left(38 + 36\sqrt[3]{2} + 27\sqrt[3]{4}\right)\left(1 - \frac{t}{t_0}\right) \\
-\frac{1944}{1574640}\sqrt{15\left(28569 + 23328\sqrt[3]{2} + 17746\sqrt[3]{4}\right)}\left(1 - \frac{t}{t_0}\right)^{3/2} \\
+\frac{648}{1574640}\left(996 + 972\sqrt[3]{2} + 749\sqrt[3]{4}\right)\left(1 - \frac{t}{t_0}\right)^{2} \\
-\frac{54}{1574640}\sqrt{15\left(37321489 + 30114648\sqrt[3]{2} + 30114648\sqrt[3]{4}\right)}\left(1 - \frac{t}{t_0}\right)^{5/2} \\
+O\left(\left(1 - \frac{t}{t_0}\right)^{4}\right).
\end{aligned}$$

Furthermore, the simplest way to proceed from here is to notice that $\mathcal{F}_0(t)/t^2$ differentiated three times is $\mathcal{R}'(t)/\mathcal{R}(t)$. This in particular means that the singularities of $\mathcal{F}_0$ are the same as the ones of R and eventually the zeros of $\mathcal{R}$. Since the zeros of $\mathcal{R}$ are only ± 1, it follows that the singularity of $\mathcal{F}_0$ is also t_0. The expansion of the third derivative of $\mathcal{F}_0(t)/t^2$ near t_0 is thus

$$\begin{aligned}
-\frac{1}{3}\sqrt{\frac{1}{5}\left(832 + 664\sqrt[3]{2} + 176\sqrt[3]{4}\right)}\Big/\sqrt{1 - \frac{t}{t_0}} - \frac{424 + 308\sqrt[3]{2} + 256\sqrt[3]{4}}{135} \\
+\frac{\sqrt{19984 + 16062\sqrt[3]{2} + 12740\sqrt[3]{4}}}{27}\sqrt{1 - \frac{t}{t_0}} - \frac{12008 + 7336\sqrt[3]{2} + 7472\sqrt[3]{4}}{3645} \\
\times\left(1 - \frac{t}{t_0}\right) \\
-\frac{\sqrt{113890440 + 92086727\sqrt[3]{2} + 76015706\sqrt[3]{4}}}{1458\sqrt{2}}\left(1 - \frac{t}{t_0}\right)^{3/2} \\
+O\left(\left(1 - \frac{t}{t_0}\right)\right).
\end{aligned}$$

Using (115), we can deduce the behavior of the coefficients of the third derivative of $\mathcal{F}_0(t)/t^2$ and then a simple exercise leads to the asymptotics of the coefficients of $\mathcal{F}_0$ from (118).

For part (2) we proceed similarly. This time,

$$\mathcal{H}(b,c) = \log(c) - c^2 - \frac{b^2}{2} - \frac{b}{2\sqrt{t}} - \frac{1}{2t}\log\left(\frac{1-b\sqrt{t}+\sqrt{(1-b\sqrt{t})^2-4tc^2}}{2}\right).$$

From the critical system satisfied by b and c eliminate b and then consider $\mathcal{R} = c^2$ to arrive at the equation satisfied by $\mathcal{R}$:

$$\begin{aligned} &144\mathcal{R}^4t^2 + \mathcal{R}^3t(60-192t) + \mathcal{R}^2(-2-52t+88t^2) \\ &\quad + \mathcal{R}(1+15t-16t^2) - (-1+2t-t^2) = 0. \end{aligned} \tag{121}$$

We are interested here in the branch which at 0 is 1. The singularity points of $\mathcal{R}$ are at the zeros of the discriminant. These are in our case the roots of

$$-11 - 128t + 41088t^2 - 20480t^3 + 4096t^4.$$

The solution we are interested in is the only solution t_0 in $(0,1)$. Approximately, $t_0 \approx 0.0180827901\ldots$. The value of $\mathcal{R}_0 = \mathcal{R}(t_0)$ can be found in terms of t_0 as

$$\mathcal{R}_0 = \frac{1}{11} + \frac{856t_0}{11} - \frac{1280t_0^2}{33} + \frac{256t_0^3}{33}.$$

Using Newton's method described in [19, VII 7] one can see that the singularity of $\mathcal{R}$ is of square root near t_0. To find the expansion, write

$$\mathcal{R}(t) = \mathcal{R}_0 + \sum_{k=1}^{M} a_k\left(1-\frac{t}{t_0}\right)^{k/2}$$

where M is the desired level of approximation. Plug this into Theorem 1.4, expand everything near t_0, match the coefficients and then solve the system thus obtained for a_k. In our case we can take for simplicity $M = 3$ and solve for a_1, a_2, a_3. The system in this case is of the form

$$\begin{cases} a_1^2v_{13} + v_{21} = 0 \\ 2a_2v_{13} + a_1^2v_{14} + v_{22} = 0 \\ 2a_3a_1v_{13} + a_2^2v_{13} + 3a_1^2a_2v_{14} + a_1^4v_{15} + a_2v_{22} + a_1^2v_{23} + v_{31} = 0 \\ 2a_4a_1v_{13} + 2a_2a_3v_{13} + 3a_1(a_2^2 + a_1a_3)v_{14} + 4a_1^3a_2v_{15} + a_3v_{22} \\ \quad +2a_1a_2v_{23} + a_1^3v_{24} + a_1v_{32} = 0, \\ 2a_5a_1v_{13} + (a_3^2 + 2a_2a_4)v_{13} + (a_2^3 + 6a_1a_2a_3 + 3a_1^2a_4)v_{14} + 6a_1^2a_2^2v_{15} \\ \quad +4a_1^3a_3v_{15} + a_4v_{22} \\ \quad +a_2^2v_{23} + 2a_1a_3v_{23} + 3a_1^2a_2v_{24} + a_1^4v_{25} + a_2v_{32} + a_1^2v_{33} = 0 \end{cases}$$

where the matrix $(v_{ij})_{i=1,3;j=1,5}$ with coefficients in $\mathbb{Q}(t_0)$ is given in reduced form by

$$\begin{bmatrix} 0 & 0 & \frac{17+64t_0-64t_0^2}{8} & 72t_0 & 144t_0^2 \\ \frac{-3595+23184t_0-10944t_0^2+2048t_0^3}{1584} & \frac{23-10903t_0+5440t_0^2-1088t_0^3}{33} & -9-16t_0+32t_0^2 & -84t_0 & -288t_0^2 \\ \frac{1769-21424t_0+12352t_0^2-2048t_0^3}{17424} & \frac{-2+631t_0-320t_0^2+64t_0^3}{33} & \frac{3-64t_0^2}{8} & 12t_0 & 144t_0^2 \end{bmatrix}.$$

There are two different solutions for a_1, a positive and a negative one. The appropriate one is the negative one in our situation because R has only non-negative coefficients (see [4] for a proof of this). Once a_1 is solved, the other coefficients are determined automatically in a unique way. Also notice here that a_1 is a square root of a number in $\mathbb{Q}(t_0)$, and that all a_k for even k are in $\mathbb{Q}(t_0)$, while a_k for k odd are in $\mathbb{Q}(t_0)/a_1$.

Now given the expansion of $\mathcal{R}$ near t_0, the rest follows as in the previous case. Namely, we can find the expansion of $\mathcal{R}'(t)/\mathcal{R}(t)$ near t_0 and thus the asymptotics of the coefficients for $\mathcal{R}'(t)/\mathcal{R}(t)$. In turn, since $\mathcal{F}_0(t)/t^2$ differentiated three times is exactly $\mathcal{R}'(t)/\mathcal{R}(t)$, the proof of (119) follows straightforwardly.

Worth mentioning is the fact that the constant C from (119) is $C = -\frac{a_1}{2\sqrt{\pi}R_0}$, which explains the square root expression of C, while the other constants d_1 and d_2 are in $\mathbb{Q}(t_0)$. □

Remark 9.3. 1. The expansion in (119) can be improved to

$$f_n \sim C\frac{(1/t_0)^n}{n^{7/2}}\left(1+\sum_{l=1}^{M}\frac{d_l}{n^l}+O\left(\frac{1}{n^{M+1}}\right)\right),$$

where C is the one from (119) and the constants d_n are actually in $\mathbb{Q}(t_0)$.

2. $\mathcal{F}_0'''$ is an algebraic function and this determines the Stokes constants in the previous result. However the algebraic equation is very long and this is the reason for not including it here. In addition, $\mathcal{F}_0$ also is the solution to some algebraic equation, though this is very long either. The differential equation satisfied by $\mathcal{F}_0$ implies a recurrence relation for the coefficients (f_n) which is again very long, thus not included.

10. Other Examples of Planar Limits

Among other computations we mention here the case of counting planar diagrams with vertices of valences 3 or 4. This corresponds to the case of potentials given by

$$\mathcal{V}_1(x) = \frac{x^2}{2} - t^{3/2}\frac{x^3}{3} - t^2\frac{x^4}{4}$$

for the counting of diagrams with a fixed number of edges. The problem of counting planar diagrams with a fixed number of faces corresponds to the potential

$$\mathcal{V}_2(x) = \frac{x^2}{2} - t^{1/2}\frac{x^3}{3} - t\frac{x^4}{4}.$$

The calculations are very similar to the ones for the extreme potentials in Sects. 8 and 9. The results are as follows. For $\mathcal{V}_1$, the asymptotics of the coefficients f_n of F_0 are given by

$$f_n = C\frac{(1/t_0)^n}{n^{7/2}}\left(1 + \frac{d_1}{n} + \frac{d_2}{n^2} + O\left(\frac{1}{n^3}\right)\right)$$

where $t = t_0$ is the closest root to $1/5$ of the polynomial equation

$$\begin{aligned} 0 = 6912 &- 13824t - 146592t^2 - 239488t^3 - 2602569t^4 - 4300752t^5 \\ &+79091888t^6 + 304167552t^7 + 410284704t^8 - 1349207040t^9 \\ &-7615156224t^{10} - 4603041792t^{11} + 31506516736t^{12}, \end{aligned}$$

and $C, d_1, d_2 \in \mathbb{Q}(t_0)$ are given numerically as

$$\begin{aligned} t_0 &= 0.2094195368\ldots \\ 1/t_0 &= 4.7751036758\ldots \\ C &= 1.4826787729\ldots \\ d_1 &= -7.2166440681\ldots \\ d_2 &= 37.5616277128\ldots. \end{aligned}$$

Similarly for the potential $\mathcal{V}_2$ we have

$$f_n = C\frac{(1/t_0)^n}{n^{7/2}}\left(1 + \frac{d_1}{n} + O\left(\frac{1}{n^2}\right)\right)$$

where t_0 is the closest root to 0.023 of the polynomial equation

$$\begin{aligned} 0 = &-43625 - 614400t + 89812992t^2 + 895478272t^3 - 3041722368t^4 \\ &-11466178560t^5 + 32248627200t^6. \end{aligned}$$

In addition, C, and $d_1 \in \mathbb{Q}(t_0)$ are numerically approximated as

$$\begin{aligned} t_0 &= 0.02305646139\ldots \\ 1/t_0 &= 43.3717899396\ldots \\ C &= 0.2023938212\ldots \\ d_1 &= -3.2617202693\ldots \end{aligned}$$

In both cases one can compute the asymptotics of the planar limit in the form

$$f_n = C\frac{(1/t_0)^n}{n^{7/2}}\left(1 + \sum_{p=1}^{M}\frac{d_p}{n^p} + O\left(\frac{1}{n^{M+1}}\right)\right)$$

for any $M \geq 1$.

11. Analyticity of the Planar Limit

In this section we prove Theorem 1.1 and some consequences. Let us introduce some notation. For a given sequence $\mathbf{a} = \{a_n\}_{n\geq 1}$ in one of the spaces $\ell^1_r(\mathbb{N})$, define

$$\alpha(\mathbf{a}) = \sup_{n\geq 1} |a_n|^{1/n}.$$

Theorem 11.1. 1. *For even potentials*

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n\geq 1} \frac{a_{2n}x^{2n}}{2n},$$

if $\alpha(\mathbf{a}) < \sqrt{8}$, then the planar limit $\mathcal{F}_0^{\mathrm{ev}}(\mathbf{a})$ is absolutely convergent as a power series in infinitely many variables. In particular $\mathcal{F}_0$ is analytic on $B^{\mathrm{ev}}_{1/\sqrt{8}}$.

2. *For the potential*

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n\geq 1} \frac{a_n x^n}{n}$$

if $\alpha(\mathbf{a}) < \sqrt{12}$, then $\mathcal{F}_0(\mathbf{a})$ is an absolutely convergent series in infinitely many variables. In particular $\mathcal{F}_0$ is analytic on $B_{1/\sqrt{12}}$.

Proof. We can write

$$\mathcal{F}_0^{\mathrm{ev}}(\mathbf{a}) = \sum_{n=1}^{\infty} \left(\sum_{\substack{\lambda\vdash 2n \\ \lambda \text{ has only even blocks}}} c_\lambda a_\lambda \right)$$

$$\mathcal{F}_0(\mathbf{a}) = \sum_{n=1}^{\infty} \left(\sum_{\lambda\vdash 2n} c_\lambda a_\lambda \right)$$

where the inner sum is over partitions of size $2n$. Note that $c_\lambda \geq 0$. Now if $|a_n| \leq r^{n/2}$, then

$$\sum_{\substack{\lambda\vdash 2n \\ \lambda \text{ has only even blocks}}} c_\lambda |a_\lambda| \leq r^n \sum_{\substack{\lambda\vdash 2n \\ \lambda \text{ has only even blocks}}} c_\lambda$$

and

$$\sum_{\lambda\vdash 2n} c_\lambda |a_\lambda| \leq r^n \sum_{\lambda\vdash 2n} c_\lambda.$$

Hence, in order to compute the radius of convergence we need to compute the radius of convergence of the planar limit $\mathcal{F}_0$ for the case of $a_n = r^{n/2}$. Similarly, for the radius of convergence of $\mathcal{F}_0^{\mathrm{ev}}$ it suffices to look at the case $a_{2n} = r^n$ and $a_{2n+1} = 0$.

Now, in these particular cases, according to Proposition 8.3, the radius of convergence of $\mathcal{F}_0^{\mathrm{ev}}$ is $1/8$ while the one of $\mathcal{F}_0$ is $1/12$. In fact, it is easy to see that the coefficient f_n of t^n in $\mathcal{F}_0^{\mathrm{ev}}(t)$ satisfies $f_n \leq 8^n$ and the coefficient f_n of t^n in $\mathcal{F}_0(t)$ satisfies $f_n \leq 12^n$. Consequently, we have that

$$\sum_{\substack{\lambda \vdash 2n \\ \lambda \text{ has only even blocks}}} c_\lambda \leq 8^n$$

and

$$\sum_{\lambda \vdash 2n} c_\lambda \leq 12^n.$$

Therefore,

$$c_\lambda \leq 8^n \text{ for any partition } \lambda \text{ of size } 2n \text{ with only even blocks}$$

and in general

$$c_\lambda \leq 12^n \text{ for any partition } \lambda \text{ of size } 2n.$$

Now, a celebrated Hardy and Ramanujan (1918) result shows that the number of partitions of size k is asymptotically $\frac{1}{4k\sqrt{3}} \mathrm{e}^{\pi\sqrt{2k/3}}(1 + o(1))$ ([2]). From this it follows easily that the series $\mathcal{F}_0^{\mathrm{ev}}$ converges for any $r < 1/8$ and $\mathcal{F}_0^{\mathrm{ev}}$ converges for any $r < 1/12$ which concludes the proof. □

Given a power series in the form

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n\geq 1} \frac{a_n x^n}{n}$$

we set

$$\alpha(\mathcal{V}) = \sup_{n\geq 1} |a_n|^{1/n}.$$

It is clear that $\mathcal{V}$ is analytic near 0 if and only if $\alpha(\mathcal{V}) < \infty$. With this notation we have the following corollary which confirms 't Hooft's conjecture. For the following statement we denote the planar limit $\mathcal{F}_0^{\mathrm{ev}}(t)$ and $\mathcal{F}_0(t)$ to be the planar limits obtained by replacing a_n by $t^{n/2} a_n$.

Corollary 11.1. *If $\mathcal{V}$ is even then $\mathcal{F}_0^{\mathrm{ev}}(t)$ has radius of convergence at least* $\frac{1}{8\sqrt{\alpha(\mathcal{V})}}$.

For arbitrary potentials $\mathcal{V}, \mathcal{F}_0(t)$ has radius of convergence $\frac{1}{12\sqrt{\alpha(\mathcal{V})}}$.

Both of these bounds are sharp.

The fact that these bounds are sharp, follow from Proposition 8.2 or Proposition 8.3. As made clear from the examples in Proposition 8.4, for the same $\alpha(\mathcal{V})$, the radius of convergence can be larger than the one given in this corollary.

Remark 11.2. The analyticity of $\mathcal{F}_0$ and $\mathcal{F}_0^{\mathrm{ev}}$ in infinitely many variables can be deduced also from the perturbation result in Theorem 12.1, though without any estimate on the radius of convergence.

Remark 11.3. The reader might wonder what happens with the planar limit if instead of considering the potentials $\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n\geq 1} \frac{a_n x^n}{n}$ we consider the potentials $\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n\geq 1} a_n x^n$. In this case the extreme potentials are given by the case where $a_n = t^{n/2}$ and this is $\mathcal{V}(x) = x^2/2 - 1/(1 - x\sqrt{t})$. It turns out after some analysis that the radius of convergence of $\mathcal{F}_0(t)$ in this case is given by t_0, which is the only solution in $(0, 1)$ of the polynomial equation

$$\begin{aligned}0 = &-226492416 + 962592768t + 34574598144t^2 + 334387408896t^3\\&+7450906184352t^4 + 21095006644064t^5 + 130097822364531t^6\\&+55792303752096t^7 + 67902575063040t^8 + 19100742451200t^9\\&+6115295232000t^{10}.\end{aligned}$$

Numerically this is approximately $t_0 = 0.04955391\ldots$. In this case, the planar limit as a function of the coefficients a_n is an analytic function on $B_{\sqrt{t_0}}$.

Remark 1.4. It would be interesting to know what happens with the case of sequences a_n which are not in ℓ^1_r. Apparently the ℓ^1_r is important for the well definition of convergent geometric series in infinitely many variables.

12. Perturbation Theory

The main result of this section is a stability result. It says that given a potential whose equilibrium measure is one interval, then, under some non-degeneracy assumptions, any small perturbation preserves the one interval support of the equilibrium measure and in addition, the planar limit depends nicely on the perturbation.

Before we state the result, we want to define a class of perturbations of a given potential V. This definition is long and depends on many parameters, however the idea is quite simple. We want to take perturbations of V so that the maximizer of the function F can be parametrized in a nice way. The reasonable way of doing this is to have perturbations close to V on some open interval containing the support of μ_{V} and large outside this open interval.

For this purpose, assume that $\mathbf{X}$ is a Banach space over the reals which will be the ambient space of the parametrization. Now, given an open subset $\mathbf{D}$ of $\mathbf{X}$ such that $\mathbf{0} \in \mathbf{D}$, and I, J open sets of $\mathbb{R}$, an integer $k \geq 1$ and $R, \delta > 0$, we define $\mathcal{U}(k, V, \mathbf{D}, I, J, R, \delta)$ the class of functions $\mathbf{V} : \mathbf{D} \times \mathbb{R} \to \mathbb{R}$ in two variables, with the properties,

(1) $\mathbf{V}(\mathbf{0}, x) = V(x)$

(2) for each $t \in \mathbf{D}$, $x \to \mathbf{V}(\mathbf{t}, x)$ satisfies (5)

(3) $(\mathbf{t}, x) \to \mathbf{V}(\mathbf{t}, x)$ is $C^{k,3}(\mathbf{D} \times \mathbb{R})$

(4) $\sup_{\mathbf{t}\in\mathbf{D}, x\in I} |\mathbf{V}(\mathbf{t}, x) - V(x)| < \delta$, & $\sup_{\mathbf{t}\in\mathbf{D}, x\in J} \|\mathrm{Hess}_x(\mathbf{V}(\mathbf{t}, \cdot) - V(\cdot))\|_{HS} < \delta$,

(5) $\inf_{\mathbf{t}\in\mathbf{D}, x\notin I} (\mathbf{V}(t, x) - 2\log|x|) \geq R$ (122)

where, $C^{k,3}$ stands for the set of jointly differentiable functions in $(\mathbf{t}, x)$ with k continuous (Fréchet) differentials in $\mathbf{t}$ and three continuous derivatives in x. Also, Hess_x stands for the Hessian with respect to the variable x and $\|\cdot\|_{HS}$ is the Hilbert–Schmidt norm.

In words, (1), (2) and (3) of (122) define the perturbation which is assumed of class $C^{k,3}$, while (4) means that the perturbation is uniformly close to V on $\mathbf{D} \times I$ while the Hessians are uniformly closed on $\mathbf{D} \times J$ and (5) encodes the fact that outside the interval I, the perturbation (minus the logarithmic term) is larger than a constant R uniformly in the parameter $\mathbf{t} \in \mathbf{D}$. We introduce here the interval J because as we will see below in the proof of Theorem 12.1, we only need the Hessians close for x on a neighborhood of the support of μ_{V}.

The reason of introducing condition (5) in (122) instead of condition (4) with $I = \mathbb{R}$ is because for large values of x, we do not need the perturbation to be close to V. We only need the perturbation to be large for large x. Actually, (4) and (5) constitute a weakening of the condition that the perturbation stays close to V uniformly on the whole $\mathbb{R}$.

Recall that we set

$$\psi_{c,b}(x) = \int_{-2}^{2} \frac{(V'(cx+b) - V'(cy+b))\mathrm{d}y}{(x-y)\pi\sqrt{4-y^2}} \quad \forall x \in [-2,2].$$

Theorem 12.1. *Assume that V is a C^3 potential satisfying (5) with $H(c,b)$ and $\psi_{c,b}$ defined by (57) and (56) respectively. Suppose that the following conditions hold true:*

$$\begin{aligned} &1. \quad (c,b) \text{ is the unique maximizer of } H;\\ &2. \quad \psi_{c,b}(x) > 0, \quad \text{for all } x \in [-2,2]. \end{aligned} \tag{123}$$

Under these assumptions, there exist

- *an interval $I \subset \mathbb{R}$,*
- *positive numbers R_0 and δ_0*

with the property that for any choice of

- *$R > R_0, 0 < \delta < \delta_0$,*
- *an open neighborhood J of $[-2c+b, 2c+b]$,*
- *a Banach space $\mathbf{X}$,*
- *and $\mathbf{V} \in \mathcal{U}(k, V, \mathbf{D}, I, J, R, \delta)$,*

the following hold

$$\begin{aligned} &(1) \quad \text{there exists an open } \mathbf{D}_0 \subset \mathbf{D} \text{ with } \mathbf{0} \in \mathbf{D}_0 \text{ and}\\ &(2) \quad (c,b) : \mathbf{D}_0 \to (0,\infty) \times \mathbb{R} \text{ which is } C^k \text{ such that } c(\mathbf{0}) = c, \quad b(\mathbf{0}) = b\\ &(3) \quad (c(\mathbf{t}), b(\mathbf{t})) \text{ is the unique maximizer of } H(\mathbf{t}, \cdot)\\ &\qquad [\text{defined by (57) for } \mathbf{V}(\mathbf{t}, \cdot)]\\ &(4) \quad \mathbf{D_0} \times [-2,2] \ni (\mathbf{t}, x) \to \psi_{c(\mathbf{t}),b(\mathbf{t})}(x) \in \mathbb{R} \text{ is positive.} \end{aligned} \tag{124}$$

Furthermore, the equilibrium measure for $V(\mathbf{t}, \cdot)$ has a single interval support for $\mathbf{t} \in \mathbf{D}_0$ and the planar limit $F_{0,\mathbf{t}} = F_{0,\mathbf{V}(\mathbf{t})}$ is a C^k function on $\mathbf{D}_0$.

In addition, if $\mathbf{X}$ *is either a finite dimensional space or of the form* $\mathbf{X} = \{(a_n)_{n\geq 1} \subset \mathbb{R} : \sum_{n\geq 1} |a_n| r^n < \infty\}$ *for some* $r > 0$ *and* V *is real analytic on a neighborhood of the support of* μ_V *such that* $(\mathbf{t}, x) \to \mathbf{V}(\mathbf{t}, x)$ *is real analytic on a neighborhood of* $\mathbf{0} \times [-2c + b, 2c + b]$*, then, we can take* $\mathbf{D}_0$ *so that* $\mathbf{t} \to c(\mathbf{t}), \mathbf{t} \to b(\mathbf{t})$ *and* $\mathbf{t} \to F_{0,\mathbf{t}}$ *are real analytic functions.*

Proof. The key point of the proof is the fact that the maximizer (c, b) of H is unique and isolated and then by perturbing a little bit the potential V, the maximizer of $H(\mathbf{t}, \cdot)$ is to be found near (c, b). Finding the maximizer $(c(\mathbf{t}), b(\mathbf{t}))$ of $H(\mathbf{t}, \cdot)$ boils down to finding the critical point of this function near (c, b). This can be achieved by the implicit function theorem and the fact that the Hessian of H is non-degenerate near (c, b).

Now technicalities. The first thing we want to do is to prove that for the unperturbed function H, (c, b) is a non-degenerate critical point. To do this we want to check that the Hessian of H at (c, b) is positive definite. For simplicity of the discussion, we will assume without any loss of generality that $c = 1$ and $b = 0$. Now the non-degeneracy is equivalent to the fact that

$$\begin{bmatrix} 2 + \int_{-2}^{2} \frac{x^2 V''(x)\mathrm{d}x}{\pi\sqrt{4-x^2}} & \int_{-2}^{2} \frac{xV''(x)\mathrm{d}x}{\pi\sqrt{4-x^2}} \\ \int_{-2}^{2} \frac{xV''(x)\mathrm{d}x}{\pi\sqrt{4-x^2}} & \int_{-2}^{2} \frac{V''(x)\mathrm{d}x}{\pi\sqrt{4-x^2}} \end{bmatrix} \tag{125}$$

is positive definite.

Recall that the critical point equations give

$$\int_{-2}^{2} \frac{xV'(x)\mathrm{d}x}{\pi\sqrt{4-x^2}} = 2 \quad \text{and} \quad \int_{-2}^{2} \frac{V'(x)\mathrm{d}x}{\pi\sqrt{4-x^2}} = 0.$$

Integrating by parts the first of these one deduces that

$$2 + \int_{-2}^{2} \frac{x^2 V''(x)\mathrm{d}x}{\pi\sqrt{4-x^2}} = 4\int_{-2}^{2} \frac{V''(x)\mathrm{d}x}{\pi\sqrt{4-x^2}}.$$

Armed with this, the non-degeneracy of the Hessian (125) follows once we prove the following

$$\int_{-2}^{2} \frac{(2 \pm x)V''(x)\mathrm{d}x}{\pi\sqrt{4-x^2}} > 0. \tag{126}$$

This follows from

$$\begin{aligned}\int_{-2}^{2} \frac{(2 \pm x)V''(x)\mathrm{d}x}{\pi\sqrt{4-x^2}} &= \int_{-2}^{2} \frac{\mathrm{d}}{\mathrm{d}x}(V'(x) - V'(\pm 2))\frac{(2 \pm x)\mathrm{d}x}{\pi\sqrt{4-x^2}} \\ &= \int_{-2}^{2} \frac{(V'(\pm 2) - V(x))\mathrm{d}x}{(\pm 2 - x)\pi\sqrt{4-x^2}} = \psi(\pm 2) > 0.\end{aligned}$$

Let $M = H(c, b)$ be the maximum of H. For any choice of $\epsilon, r > 0$ with $r > \epsilon > 0$, obviously one has

$$\sup\{H(u, v) :\ u > 0, v \in \mathbb{R}, r^2 > (u - c)^2 + (v - b)^2 \geq \epsilon^2\} < M. \quad (127)$$

Indeed if this is not the case, then there is a sequence (c_n, b_n) such that $r^2 > (u_n - c)^2 + (v_n - b)^2 > \epsilon^2$ so that $\lim_{n\to\infty} H(u_n, v_n) = M$. Passing eventually on subsequences, we may assume that u_n and v_n converge to u and v. Clearly $u \neq 0$, otherwise $H(u, v) = -\infty$. This implies that $H(u, v) = M$ and at the same time, (u, v) is within positive distance from (c, b), hence contradicting the uniqueness of the maximizer.

Next, consider

$$U(x) := V(x) - 2 \log |x|,$$

and notice that from (5) we have $U(x) \geq C > -\infty$ and $\lim_{|x|\to\infty} U(x) = +\infty$. Now we use (37) to justify that

$$H(u, v) = -\int_{-2}^{2} \frac{U(ux + v)\mathrm{d}x}{2\pi\sqrt{4 - x^2}} - \int_{-2}^{2} \frac{\log|x + v/u|}{2\pi\sqrt{4 - x^2}} \leq -\int_{-2}^{2} \frac{U(ux + v)\mathrm{d}x}{2\pi\sqrt{4 - x^2}}.$$

Assuming that $(u-c)^2 + (v-b)^2 \geq r^2$, it is easy to deduce that, $u + |v| \geq \sqrt{r^2/2 - c^2 - b^2}$, and thus,

$$\begin{aligned} H(u, v) &\leq -\int_{-1}^{1} \frac{U(ux + v)\mathrm{d}x}{2\pi\sqrt{4 - x^2}} - \int_{1}^{2} \frac{U(ux + v)\mathrm{d}x}{2\pi\sqrt{4 - x^2}} - \int_{-2}^{-1} \frac{U(ux + v)\mathrm{d}x}{2\pi\sqrt{4 - x^2}} \\ &\leq -C/3 - h(\sqrt{r^2/2 - c^2 - b^2}) \end{aligned} \quad (128)$$

where $h(x) = \inf_{|y|\geq x} U(y)/6$. In particular, for large r we learn that $H(u, v) < M$.

Equations (127) and (128) guarantee that for any $\epsilon > 0$, there exists $\delta_0 \in (0, 1)$ such that

$$H(u, v) < M - 3\delta_0 \quad (129)$$

for all (u, v) outside a ball of radius ϵ around (c, b). We take $R_0 > 0$ such that $|x| \geq R_0$ implies $h(x) > -C/3 - M + 3$ and define $I = [-R_0, R_0] \cup [-2c - 1 + b, 2c + 1 + b]$. The purpose of this choice of I is to make it a neighborhood of the support of μ_{V}.

With these choices, for any $R > R_0, 0 < \delta < \delta_0$ and $\mathbf{V} \in \mathcal{U}(k, V, \mathbf{D}, I, J, R, \delta)$, from the conditions (4) and (5) of (122), and the reasoning which led to (128), one gets for $r = \sqrt{2(R_0^2 + b^2 + c^2)}$ that

1. $|H(\mathbf{t}, u, v) - H(u, v)| < \delta, \quad$ for $r^2 > (u - c)^2 + (v - b)^2 > \epsilon^2$
2. $H(\mathbf{t}, u, v) < M - 3 \quad$ for $(u - c)^2 + (v - b)^2 > r^2$.

We are led to the conclusion that for all $\mathbf{t} \in \mathbf{D}, \max_{u>0,v\in\mathbb{R}}\{H(\mathbf{t}, u, v)\}$ is attained for (u, v) in the ball of radius ϵ around (c, b). Indeed, otherwise, assume that there is a maximizer (u, v) outside the ball $B_\epsilon(c, b)$. Since, $|H(\mathbf{t}, u, v) - H(u, v)| < \delta$, combined with (129), implies that $H(\mathbf{t}, u, v) <$

$M - 2\delta$. This contradicts (4) of Eq. (122) from which we gather that $H(\mathbf{t}, c, b) - H(c, b) > -\delta$, or $H(\mathbf{t}, c, b) > M - \delta > H(\mathbf{t}, u, v) + \delta$, thus (u, v) can not be a maximizer of $H(\mathbf{t}, \cdot)$.

The maximizer is a critical point of $H(\mathbf{t}, u, v)$, therefore $\nabla_{u,v} H(\mathbf{t}, \cdot) = 0$. To solve for (u, v), we interpret it as the definition of an implicit function $\mathbf{t} \to (c(\mathbf{t}), b(\mathbf{t}))$. This can be done thanks to the combination of the last part of (4) of (122), (1) of (123) and the implicit function theorem. These yield for a set $\mathbf{D}_0 \subset \mathbf{D}$, which contains $\mathbf{0}$ that there exists a C^k function $\mathbf{t} \to (c(\mathbf{t}), b(\mathbf{t}))$ which is the maximizer of $H(\mathbf{t}, \cdot)$. Taking a smaller subset of $\mathbf{D}_0$, it is easy to show that $\psi_{c(\mathbf{t}),b(\mathbf{t})} > 0$ on $[-2, 2]$ and the C^k dependence of $F_{0,\mathbf{t}}$ on $\mathbf{t}$ is a simple consequence of (51).

In the case of analytic perturbations with $\mathbf{X}$ a finite dimensional space, the only thing we need to point out is that (cf. [25]) the implicit function theorem produces analytic versions $c(\mathbf{t})$ and $b(\mathbf{t})$ for $\mathbf{t}$ in an eventually smaller $\mathbf{D}_0$. The analyticity of $F_{0,\mathbf{t}}$ follows from (51).

On the other hand in the case $\mathbf{X} = \{(a_n)_{n\geq 1} \subset \mathbb{R} : \sum_{n\geq 1} |a_n| r^n < \infty\}$, one needs a bit more work. The analyticity of functions in infinitely many variables is trickier than the case of analytic functions in finitely many variables. However, our space here is essentially $\ell^1(\mathbb{N})$ over the real numbers and for this case many things are like in the finite dimensional cases.

What we mean here is that for the case of $\ell^1(\mathbb{N})$ over the complex numbers, the theory of analytic functions is treated in [26] and [36]. The main results are that every holomorphic function on $\ell^1(\mathbb{N})$ has a power series expansion and every absolutely convergent power series expansion defines a holomorphic function.

In our situation, the functions are real analytic (meaning they have a power series expansion), thus by complexification they become complex analytic and therefore they are holomorphic functions. Then, for the complexification, we know that the implicit function theorem yields that the resulting functions $c(\mathbf{t}), b(\mathbf{t})$ and $F_{0,\mathbf{t}}$ all are smooth functions of $\mathbf{t}$ on a small neighborhood of $\mathbf{X}$. Furthermore, since F is actually a holomorphic function it is not hard to prove that the choices of $c(\mathbf{t}), b(\mathbf{t})$ and $F_{0,\mathbf{t}}$ can be made holomorphic. Using this we can conclude that the real parts of $c(\mathbf{t}), b(\mathbf{t})$ and $F_{0,\mathbf{t}}$ are real analytic. □

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Appendix A. The First Few Terms of $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{F}_0$

In this appendix we give the first few terms of the unique solution $(R, S) \in \mathcal{A}$ of Eq. (13) and also of the formal planar limit $\mathcal{F}_0$.

$$\mathcal{S} = a_1 + a_1 a_2 + 2a_3 + a_1 a_2^2 + a_1^2 a_3 + 4a_2 a_3 + 6a_1 a_4 + 6a_5 + a_1 a_2^3 + 3a_1^2 a_2 a_3 \\ +6a_2^2 a_3 + 8a_1 a_3^2 + a_1^3 a_4 + 18 a_1 a_2 a_4 + 18 a_3 a_4 + 12 a_1^2 a_5 + 18 a_2 a_5$$

$$
\begin{aligned}
&+30a_1a_6+20a_7+a_1a_2^4+6a_1^2a_2^2a_3+8a_2^3a_3+2a_1^3a_3^2+32a_1a_2a_3^2+12a_3^3\\
&+4a_1^3a_2a_4+36a_1a_2^2a_4+42a_1^2a_3a_4+72a_2a_3a_4+54a_1a_4^2+a_1^4a_5\\
&+48a_1^2a_2a_5+36a_2^2a_5+108a_1a_3a_5+72a_4a_5+20a_1^3a_6+120a_1a_2a_6\\
&+80a_3a_6+90a_1^2a_7+80a_2a_7+140a_1a_8+70a_9+O(a^{11})\\
\mathcal{R}=&\,1+a_2+a_2^2+2a_1a_3+3a_4+a_2^3+6a_1a_2a_3+4a_3^2+3a_1^2a_4+9a_2a_4\\
&+12a_1a_5+10a_6+a_2^4+12a_1a_2^2a_3+6a_1^2a_3^2+16a_2a_3^2+12a_1^2a_2a_4+18a_2^2a_4\\
&+42a_1a_3a_4+18a_4^2+4a_1^3a_5+48a_1a_2a_5+36a_3a_5+30a_1^2a_6+40a_2a_6\\
&+60a_1a_7+35a_8+a_2^5+20a_1a_2^3a_3+30a_1^2a_2a_3^2+40a_2^2a_3^2+32a_1a_3^3\\
&+30a_1^2a_2^2a_4+30a_2^3a_4+20a_1^3a_3a_4+210a_1a_2a_3a_4+84a_3^2a_4+63a_1^2a_4^2\\
&+90a_2a_4^2+20a_1^3a_2a_5+120a_1a_2^2a_5+132a_1^2a_3a_5+180a_2a_3a_5+252a_1a_4a_5\\
&+72a_5^2+5a_1^4a_6+150a_1^2a_2a_6+100a_2^2a_6+260a_1a_3a_6+150a_4a_6+60a_1^3a_7\\
&+300a_1a_2a_7+160a_3a_7+210a_1^2a_8+175a_2a_8+280a_1a_9+126a_{10}\\
&+O(a^{11})\\
\mathcal{F}_0=&\,\frac{a_1^2}{2}+\frac{a_2}{2}+\frac{1}{2}a_1^2a_2+\frac{a_2^2}{4}+\frac{1}{2}a_1^2a_2^2+\frac{a_2^3}{6}+\frac{1}{2}a_1^2a_2^3+\frac{a_2^4}{8}+\frac{1}{2}a_1^2a_2^4+\frac{a_2^5}{10}\\
&+a_1a_3+\frac{1}{3}a_1^3a_3+2a_1a_2a_3+a_1^3a_2a_3+3a_1a_2^2a_3+2a_1^3a_2^2a_3+4a_1a_2^3a_3\\
&+\frac{2a_3^2}{3}+2a_1^2a_3^2+\frac{1}{2}a_1^4a_3^2+2a_2a_3^2+8a_1^2a_2a_3^2+4a_2^2a_3^2+4a_1a_3^3+\frac{a_4}{2}\\
&+\frac{3}{2}a_1^2a_4+\frac{1}{4}a_1^4a_4+a_2a_4+\frac{9}{2}a_1^2a_2a_4+a_1^4a_2a_4+\frac{3}{2}a_2^2a_4+9a_1^2a_2^2a_4+2a_2^3a_4\\
&+6a_1a_3a_4+7a_1^3a_3a_4+24a_1a_2a_3a_4+6a_3^2a_4+\frac{9a_4^2}{8}+9a_1^2a_4^2+\frac{9}{2}a_2a_4^2\\
&+2a_1a_5+2a_1^3a_5+\frac{1}{5}a_1^5a_5+6a_1a_2a_5+8a_1^3a_2a_5+12a_1a_2^2a_5+3a_3a_5\\
&+18a_1^2a_3a_5+12a_2a_3a_5+18a_1a_4a_5+\frac{18a_5^2}{5}+\frac{5a_6}{6}+5a_1^2a_6+\frac{5}{2}a_1^4a_6\\
&+\frac{5a_2a_6}{2}+20a_1^2a_2a_6+5a_2^2a_6+20a_1a_3a_6+6a_4a_6+5a_1a_7+10a_1^3a_7\\
&+20a_1a_2a_7+8a_3a_7+\frac{7a_8}{4}+\frac{35}{2}a_1^2a_8+7a_2a_8+14a_1a_9+\frac{21a_{10}}{5}+O(a^{11})
\end{aligned}
$$

where each a_k is given the degree k. For example the monomial $a_1^2a_2^2a_4$ has degree $2\times 1+2\times 2+4=10$. The meaning of $O(a^{11})$ is that the degree of the remaining terms is at least 11.

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